MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 6, N° 1, 1977, pp. 51—56

A CHARACTERISATION OF CHEBYSHEVIAN SUBSPACE OF $Y^{\perp}$ — TYPE

by

COSTICĂ MUSTĂŢA
(Cluj-Napoca)

1. Let be given a real linear space Z. For any nonvoid set E we denote by Z^E the linear space of all functions from E to Z with the operations of addition and multiplication by real scalars defined pointwisely.

Consider now two nonvoid sets X, Y such that $Y \subseteq X$ and two normed linear subspace M_X and M_Y of Z^X, respectively of Z^Y, such that $f|_Y \in M_Y$ for all $f \in M_X$, where $f|_Y$ denotes the restriction of f to Y. Denote by $\| \|_X$ and $\| \|_Y$ the norms on M_X, respectively M_Y.

Definition 1. *We say that the norm* $\| \|_Y$ *is compatible with the norm* $\| \|_X$ *if*

$$\|f|_Y\|_Y \leq \|f\|_X, \tag{1}$$

for all $f \in M_X$.

In the sequel, the norms $\| \|_X$ and $\| \|_Y$ will considered always compatible.

Let $K_X \subseteq M_X$ and $K_Y \subseteq M_Y$ be two convex cones with the vertex in the origin of M_X, respectively M_Y such that $f|_Y \in K_Y$, for all $f \in K_X$.

Definition 2. *We say that* K_Y *is a* P—*cone if for all* $f \in K_Y$ *there exists* $F \in K_X$ *such that*

1) $f = F|_Y$,

2) $\|f\|_Y = \|F\|_X$.

If further, the function F *with the properties* 1) *and* 2) *is unique,* K_Y *is called* PU—*cone. The function* F *is called an extension of* f.

2. Let

$$X_K = K_X - K_X, \tag{2}$$

be the linear subspace of M_X, generated by the cone K_X and

$$Y^{\perp}_{X_K} = \{g : g \in X_K,\ g|_Y = \theta_Y\}, \tag{3}$$

where θ_Y denotes the zero function in M_Y, i.e. $\theta_Y(y) = 0$, for all $y \in Y$.

Definition 3. *We say that the subspace $Y^{\perp}_{X_K}$ is K_K — proximinal if for all $f \in K_X$ there exists an element $g_0 \in Y^{\perp}_{X_K}$ such that*

$$\|f - g_0\|_X = d(f, Y^{\perp}_{X_K}) = \inf\{\|f - g\|_X : g \in Y^{\perp}_{X_K}\}. \tag{4}$$

If further, for all $f \in K_X$ there exists a unique $g_0 \in Y^{\perp}_{X_K}$ such that the equality (4) *holds, then $Y^{\perp}_{X_K}$ is called K_X—Chebyshveian. An element $g_0 \in Y^{\perp}_{X_K}$ such that $\|f - g_0\|_X = d(f, Y^{\perp}_{X_K})$ is called an element of best approximation of f by elements of $Y^{\perp}_{X_K}$.*

3. The following two theorems show that the best approximation properties of the subspace $Y^{\perp}_{X_K}$ in M_X are connected with the extension properties of K_Y.

THEOREM 1. *If K_Y is a P—cone then:*

(*a*) *for all $f \in K_X$, the following equality holds*

$$\|f|_Y\| = d(f, Y^{\perp}_{X_K}); \tag{5}$$

(*b*) *for every $f \in K_X$, the elements of best approximation of f by elements of $Y^{\perp}_{X_K}$ are exactly the elements of the form $f - F$, where F is an extension of $f|_Y$.*

Proof. (*a*) For $g \in Y^{\perp}_{X_K}$ we have:

$$\|f|_Y\|_Y = \|f|_Y - g|_Y\|_Y = \|(f-g)|_Y\|_Y \leqslant \|f-g\|_X,$$

such that $\|f|_Y\|_Y \leqslant d(f, Y^{\perp}_{X_K})$.

On the other hand,

$$\|f|_Y\|_Y = \|f - (f-F)\|_X \geqslant \inf\{\|f-g\|_X : g \in Y^{\perp}_{X_K}\} = d(f, Y^{\perp}_{X_K}),$$

where F is an extension of $f|_Y$ to X. Therefore, the equality (5) holds.

(*b*) If $f \in K_X$ and $g \in Y^{\perp}_{X_K}$ is an element of best approximation of f, then by (5), $\|f-g\|_X = d(f, Y^{\perp}_{X_K}) = \|f|_Y\|_Y$ and $(f-g)|_Y = f|_Y$. It follows

that $f - g$ is an extension of $f|_Y$ to X. The fact that $f - F$ is a best approximation of f by elements of $Y^{\perp}_{X_K}$, for every extension F of $f|_Y$ to X, follows by the equalities:

$$d(f, Y^{\perp}_{X_K}) = \|f|_Y\|_Y = \|f - (f - F)\|_X.$$

THEOREM 2. (a) *If K_Y is a P—cone, then $Y^{\perp}_{X_K}$ is K_X—proximinal;*

(b) *If K_Y is a P—cone, then $Y^{\perp}_{X_K}$ is K_X—Chebyshevian if and only if K_Y is a PU—cone.*

Proof. The theorem follows from theorem 1 (b).

If $K_Y = M_Y$ and K_Y is P — cone, respectively PU — cone, then M_Y is called *P — space*, respectively *PU — space*.

Let us denote by $Y^{\perp}$, the following subspace of M_X:

$$Y^{\perp} = \{f : f \in M_X,\ f|_Y = \theta_Y\}. \tag{6}$$

Then, the theorems 1 and 2 become:

THEOREM 3. *If M is P — space, then:*

(*a*) *for all $f \in M_X$, the following equality holds:*

$$\|f|_Y\|_Y = d(f, Y^{\perp}); \tag{7}$$

(*b*) *for every $f \in M_X$, the elements of best approximation of f by elements of $Y^{\perp}$ are exactly the elements of the form $f - F$, where F is an extension of $f|_Y$ to X.*

THEOREM 4. (*a*) *If M_Y is a P — space, then $Y^{\perp}$ is proximinal;*

(*b*) *If M_Y is a P — space, then $Y^{\perp}$ is Chebyshevian if and only if M_Y is a PU — space.*

For the definition of proximinal and Chebyshevian sets see [9].

4. We shall give some particular cases of the above theorems.

I. If X is a normed linear space, Y a linear subspace of X, X^* the conjugate space of X, Y^* the conjugate space of Y, then by the Hahn-Banach theorem, Y^* is a P — space. In this case, theorem 3 (*a*) and theorem 4 (*b*) were proved by R. R. PHELPS [8].

II. For a metric space (X, d), a subset Y of X and a fixed element x_0 of Y, let

$$\mathrm{Lip}_0\, X = \{f : \ f : X \to \mathbf{R},\ \sup_{\substack{x \neq y \\ x, y \in X}} \frac{|f(x) - f(y)|}{d(x, y)} < \infty,\ f(x_0) = 0\}, \tag{8}$$

$$\mathrm{Lip}_0\, Y = \{h : \ h : Y \to \mathbf{R},\ \sup_{\substack{x \neq y \\ x, y \in Y}} \frac{|h(x) - h(y)|}{d(x, y)} < \infty,\ h(x_0) = 0\}, \tag{9}$$

be the linear space of Lipschitz functions on X, respectively Y, which vanish on x_0, with the norms

(10) $$\|f\|_X = \sup \{|f(x) - f(y)|/d(x, y) : x \neq y,\ x, y \in X\},$$

(11) $$\|h\|_Y = \sup \{|h(x) - h(y)|/d(x, y) : x \neq y,\ x, y \in Y\}.$$

By a theorem of S. BANACH [1], rediscovered by J. CZIPSER and L. GÉHER [2], the space $\mathrm{Lip}_0 Y$ is a P-space with respect to $\mathrm{Lip}_0 X$. In this case, theorems 3 and 4 were proved in [5].

III. A topological space is called extremally disconnected if the closure of every open set is open. If Ω is a compact Hausdorff space, denote by $C(\Omega)$ the Banach space of all continuous real functions defined on Ω with the sup-norm.

Let Ω be an extremally disconnected compact Hausdorff space, X a Banach space, Y a subspace of X. By a theorem of L. NACHBIN [6] $L(Y, C(\Omega))$ is a P-space in $L(X, C(\Omega))$, so that theorems 3 and 4 can be applied. Here $L(E, F)$ denotes the space of all continuous linear operators between the Banach spaces E and F.

IV. Let (X, d) be a metric linear space, d being a invariant metric for translation, i.e. $d(x, y) = d(x - y, \theta)$. Let

(12) $$S_X^\circ = \{f :\ f : X \to \mathbf{R},\ \sup \{|f(x)|/d(x, \theta) : x \neq \theta,\ x \in X\} < \infty,$$
$$f(\theta) = 0,\ f(x + y) \leqslant f(x) + f(y),\ x, y \in X\},$$

be the cone defined by G. PANTELIDIS [7].

For a subspace Y of X, the cone S_Y° is defined in a similar way.

It was proved in [5], that S_X° is a convex cone in $\mathrm{Lip}_0 X$, S_Y° is a convex cone in $\mathrm{Lip}_0 Y$ and S_Y° is a P-cone.

Let

(13) $$X_S = S_X^\circ - S_X^\circ,$$

be the linear space generated by the cone S_X°. In this case, theorem 1 and theorem 2 were proved in [5].

V. If X is a normed linear space, Y a nonvoid convex subset of X such that $\theta \in Y$, put

(14) $$C_X = \{f : f \in \mathrm{Lip}_0 X,\ f \text{ is convex}\},$$

(15) $$C_Y = \{h : h \in \mathrm{Lip}_0 Y.\ h \text{ is convex}\}.$$

Then C_Y is a P-cone and theorem 1 and theorem 2 can be applied.

5. In this section we intend to study the relation between the extremal elements of the unit ball of M_Y and the faces of the unit ball of M_X (the notation are as in section 1.).

If $(E, \|\ \|)$ is a normed space, denote by B_E and S_E the unit ball, respectively the unit sphere of E, i.e.

(16) $$B_E = \{x \in E : \|x\| \leqslant 1\},$$
$$S_E = \{x \in E : \|x\| = 1\}.$$

An *extremal element* of a convex set C in a linear space E is an element $x \in C$ such that $\lambda x_1 + (1 - \lambda) x_2 = x$ for $x_1, x_2 \in C$ and $\lambda \in (0, 1)$ implies $x_1 = x = x_2$.

A *face* of the unit ball B_E is a convex subset F of S_E such that $\lambda x_1 + (1 - \lambda) x_2 \in F$ for $x_1, x_2 \in B_E$ and $\lambda \in (0, 1)$ implies that $x_1, x_2 \in F$. Obviously, a face which contain exactly one element is an extremal element of B_E.

For $h \in M_Y$, denote by

(17) $$P_Y(h) = \{f : f \in M_X,\ f|_Y = h,\ \|f\|_X = \|h\|_Y\},$$

the set of all extension of h.

Then $P_Y(h)$ is a nonvoid, convex, bounded and closed subset of M_X.

THEOREM 5. *An element $h \in B_{M_Y}$ is an extremal element of B_{M_Y} if and only if $P_Y(h)$ is a face of B_{M_X}.*

Proof. Suppose h is an extremal element of B_{M_Y}. Let $\lambda \in (0, 1)$ and $f_1, f_2 \in B_{M_X}$ be such that $\lambda f_1 + (1 - \lambda) f_2 \in P_Y(h)$. Then $\lambda f_1|_Y + (1 - \lambda) f_2|_Y = h$, and since h is an extremal element of B_{M_Y}, it follows that $f_1|_Y = f_2|_Y = h$, so that $\|f_1|_Y\|_Y = \|f_2|_Y\|_Y = \|h\|_Y = 1$. Since the norms $\|\ \|_X$ and $\|\ \|_Y$ are supposed compatible (see definition 1.) it follows that $\|f_1\|_X = \|f_2\|_X = 1$. We proved that $f_1, f_2 \in P_Y(h)$ which shows that $P_Y(h)$ is a face of B_{M_X}.

Conversely, suppose h is not an extremal element of B_{M_Y}. Then there exist two elements $h_1, h_2 \in B_{M_Y}$, $h_1 \neq h$, $h_2 \neq h$ and $\lambda \in (0, 1)$ such that $\lambda h_1 + (1 - \lambda) h_2 = h$. Let $f_1' \in P_Y(h_1)$ and $f_2' \in P_Y(h_2)$. Then $\lambda f_1'|_Y + (1 - \lambda) f_2'|_Y = h$ and $1 = \|\lambda f_1'|_Y + (1 - \lambda) f_2'|_Y\|_Y \leqslant \|\lambda f_1' + (1 - \lambda) f_2'\|_X \leqslant 1$, so that $\lambda f_1' + (1 - \lambda) f_2' \in P_Y(h)$. But f_1' and f_2' do not belong to $P_Y(h)$ since $f_1'|_Y \neq h$ and $f_2'|_Y \neq h$, so that $P_Y(h)$ is not a face of B_{M_X}.

Suppose now, $\mathrm{Lip}_0 X$ and $\mathrm{Lip}_0 Y$ be as in the case II. from section 4. If $h \in \mathrm{Lip}_0 Y$, then the functions

(18) $$f_1(x) = \inf \{h(y) + \|h\|_Y\, d(x, y) : y \in Y\},\ x \in X,$$
$$f_2(x) = \sup \{h(y) - \|h\|_Y\, d(x, y) : y \in Y\},\ x \in X$$

are extensions of h (see [4]) and further, they are extremal elements of the set $P_Y(h)$.

Indeed, one can prove that

(19) $$f_2(x) \leqslant f(x) \leqslant f_1(x),\quad x \in X,$$

for all $f \in P(h)_Y$ (see [5]). If $\varphi, \psi \in P_Y(h)$ and $\lambda \in (0, 1)$ are such that $\lambda\varphi + (1 - \lambda)\psi = f_1$, then

(20) $$0 \leqslant \lambda(f_1 - \varphi) = (1 - \lambda)(\psi - f_1),$$

and by (19) it follows $\varphi = \psi = f_1$, so that f_1 is an extremal element of $P_Y(h)$. In a similar way one can show that f_2 is an extremal element of $P_Y(h)$.

Since, by theorem 5, h is an extremal element of $B_{\mathrm{Lip}_0 Y}$ if and only if $P_Y(h)$ is a face of $B_{\mathrm{Lip}_0 X}$, and an extremal element of a face of the unit ball of a normed linear space is an extremal element of the ball, it follows:

If h is an extremal element of the unit ball of $\mathrm{Lip}_0 Y$, *then the functions* f_1, f_2 *defined by the formulae* (18) *are extremal elements of the unit ball of* $\mathrm{Lip}_0 X$.

REFERENCES

[1] Banach, S., *Wstep to teorii funkji rzeczywistych*, Warszawa—Wroclaw, 1951.

[2] Czipser, J., Géher, L., *Extension of function satisfying a Lipschitz condition*, Acta Math. Acad. Sci. Hungar., **6**, 213—220, (1955).

[3] Kolumbán, I., *Ob edinstvenosti prodolzenia lineinyh funktionalov*, Mathematica (Cluj), **4** (27), 267—270, (1962).

[4] Mustăța, C., *Asupra unor subspații cebîșeviene din spațiul normat al funcțiilor lipschitziene*, Rev. Anal. Num. Teor. Aprox., **2**, 1, 81—87, (1973).

[5] — , *O proprietate de monotonie a operatorului de cea mai bună aproximare în spațiul funcțiilor lipschitziene*, Rev. Anal. Num. Teor. Aprox., **3**, 2, 153—160, (1974).

[6] Nachbin, L., *A theorem of the Hahn-Banach type for linear transformation*, Trans. Amer. Math. Soc., **68**, 28—46, (1950).

[7] Pantelidis, G., *Approximationstheorie für metrische lineare Räume*, Math. Ann., **184**, 30—48, (1969).

[8] Phelps, R. R., *Uniqueness of Hahn-Banach extension and unique best approximation*, Trans. Amer. Math. Soc., **25**, 238—255, (1960).

[9] Singer, I., *Cea mai bună aproximare în spații vectoriale normate prin elemente din subspații vectoriale*, Ed. Acad. R.S.R., București, 1967.

Received 16. II. 1977.

Universitatea „Babeș—Bolyai"
Cluj-Napoca
Institutul de matematică