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A REMARK ON TCHEBYCHEFF SYSTEMS

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One of the very useful properties of the Tchebycheff systems is the following:

THEOREM 1. *Let* $\{u_0, u_1, \ldots, u_{2n}\}$ *be a Tchebycheff system on* $[a, b]$. *If* $x_1, \ldots, x_n$ *are distinct points in* $[a, b]$, *then there exist* $a_0, a_1, \ldots, a_{2n} \in R$ *such that, if we put* $F = \sum_{i=0}^{2n} a_i u_i$, *then*

1) $F(x_i) = 0,\ i = 1, \ldots, n,$
2) $F(x) > 0,\ x \in [a, b],\ x \neq x_1, \ldots, x_n.$

This property is essentially employed in many proofs, and it is interesting to know if it is characteristic or not for the Tchebycheff systems. We shall answer this question in the second part of the paper. In the first part we deduce from the theorem 1 a result concerning the separation of certain point sets in the plane.

1. THEOREM 2. *Let* $\{u_0, \ldots, u_{2n}\}$ *be a Tchebycheff system on* $[a, b]$, $f: [a, b] \to R$ *an upper — semicontinuous function,* $x_1, \ldots, x_n$ *distinct points in* $[a, b]$, *and* $y_i > f(x_i),\ i = 1, \ldots, n$. *Then there exists a linear combination F of the functions* $u_0, \ldots, u_{2n}$ *such that:*

a) $F(x_i) = y_i,\ i = 1, 2, \ldots, n,$
b) $f(x) < F(x),\ x \in [a, b].$

Proof. We denote by $\mathrm{Sp}(u_0, u_1, \ldots, u_{2n})$ the set of all linear combinations of the functions $u_0, \ldots, u_{2n}$. Consider $P, Q \in \mathrm{Sp}(u_0, \ldots, u_{2n})$ such that $P(x_i) = y_i,\ i = 1, 2, \ldots, n,$ and

1) $Q(x_i) = 0,\ i = 1, 2, \ldots, n.$
2) $Q(x) > 0,\ x \in [a, b],\ x \neq x_1, \ldots, x_n.$

If we put $g = f - P$, then g is upper — semicontinuous and $g(x_i) = f(x_i) - y_i < 0$. There exists an open neighbourhood V_i of x_i such that if $x \in V_i$ then $g(x) < 0$. Consider the compact set $T = [a, b] \setminus \bigcup_{i=1}^{n} V_i$. The function g is upper bounded on T by a number M. Q is continuous and positive on T. If $m = \min \{Q(t) : t \in T\}$ then $m > 0$.

Let us consider the number $k > 0$ such that $km > M$. We have $F = P + kQ \in Sp(u_0, \ldots, u_{2n})$ and F satisfies a) and b).

2. Now we prove the converse of the theorem 1.

THEOREM 3. *Consider the functions* $u_0, \ldots, u_{2n} \in C[a, b]$ *such that for any distinct points* $x_1, \ldots, x_n$ *in* $[a, b]$ *there exists* $F \in Sp(u_0, \ldots, u_{2n})$ *which satisfies:*

1) $F(x_i) = 0,\ i = 1, \ldots, n,$
2) $F(x) > 0,\ x \in [a, b],\ x \neq x_1, \ldots, x_n.$

Then $\{u_0, \ldots, u_{2n}\}$ *is a Tchebycheff system on* $[a, b]$.

Proof. Assume that $\{u_0, \ldots, u_{2n}\}$ is not a Tchebycheff system on $[a, b]$. Then there exist the distinct points $t_0, \ldots, t_{2n}$ in $[a, b]$ such that $|u_i(t_j)| = 0$, $i,\ j = 0, 1, \ldots, 2n$.

This means that one column of the determinant is a linear combination of the others. Let

$$(1) \qquad u_i(t_0) = \sum_{j=1}^{2n} c_j u_i(t_j), \quad i = 0, 1, \ldots, 2n.$$

It is easy to prove that if $F \in \mathrm{Sp}(u_0, \ldots, u_{2n})$ then

$$(2) \qquad F(t_0) = \sum_{j=1}^{2n} c_j F(t_j).$$

The numbers c_j do not all vanish. Indeed, if $c_1 = \ldots = c_{2n} = 0$, then $u_i(t_0) = 0$; $i = 1, \ldots, 2n$ and $F(t_0) = 0$ for any $F \in \mathrm{Sp}(u_0, \ldots, u_{2n})$. By choosing $x_i \neq t_0$, $i = 1, 2, \ldots, n$, we contradict the assumptions of the theorem. Without restricting the generality we may assume that the coefficients $c_1, \ldots, c_h$ are negative and the others nonnegative. Here $0 \leq h \leq 2n$. There are three cases to be considered:

a) $h \leq n-1$. There exists $F \in \mathrm{Sp}(u_0, \ldots, u_{2n})$ such that F vanishes on $t_0, t_1, \ldots, t_h$ and it is positive on $t_{h+1}, \ldots, t_{2n}$. We obtain, because of (2).

$$c_{h+1} = \ldots = c_{2n} = 0.$$

This means that $F(t_0) = \sum_{j=1}^{h} c_j F(t_j)$ for all $F \in \mathrm{Sp}(u_0, \ldots, u_{2n})$.

Consider $F_1 \in \mathrm{Sp}(u_0, \ldots, u_{2n})$ which vanishes on t_0 and it is positive for $t_1, \ldots, t_h$. Then

$$\sum_{j=1}^{h} c_j F_1(t_j) = 0, \quad c_j < 0, \quad F_1(t_j) > 0, \quad j = 1, \ldots, h.$$

This is a contradiction.

b) $h = n$. In this case it is easy to prove that the coefficients $c_{n+1}, \ldots, c_{2n}$ do not all vanish. Let $c_{2n} > 0$. It follows from (1)

$$u_i(t_{2n}) = \frac{1}{c_{2n}} u_0(t_0) + \sum_{j=1}^{2n-1} \left(-\frac{c_j}{c_{2n}}\right) u_i(t_j),$$

$i = 0, 1, \ldots, 2n$.

Now the coefficients $\frac{1}{c_{2n}}, -\frac{c_1}{c_{2n}}, \ldots, -\frac{c_n}{c_{2n}}$ are positive, and the number h' of the negative coefficients is at most $n-1$. Thus, this case can be reduced to the first one.

c) $h \geqslant n + 1$. Consider $F \in \mathrm{Sp}(u_0, \ldots, u_{2n})$ which vanishes on t_0, $t_{h+1}, \ldots, t_{2n}$ and it is positive for $t_1, \ldots, t_h$. From (2):

$$\sum_{j=1}^{h} c_j F(t_j) = 0.$$

But $c_j < 0$, $F(t_j) > 0$, $j = 1, \ldots, h$, and we find a new contradiction.

The theorem 3 is herewith proved. Consequently the property expressed in theorem 1 is characteristic for the Tchebycheff systems.

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