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ON A FIXED POINT THEOREM IN A SET WITH TWO METRICS

by

IOAN A. RUS

(Cluj-Napoca)

0. Introduction. Let X be a nonempty set, d and δ two metrics on X and $f:X \to X$ a mapping. For such mappings, M. G. MAIA [11] proved a fixed point theorem which was generalized in many directions by K. ISÉKI ([7], [8], [9]), I. A. RUS [17], S. P. SINGH [20].

The following theorem given in [17] is an useful theorem for the applications.

THEOREM 1. *If*

(i) $\exists c > 0 : d(f(x), f(y)) \leqslant c\delta(x, y), \ \forall x, y \in X$,

(ii) (X, d) *is a complete metric space,*

(iii) $f: (X, d) \to (X, d)$ *is continuous,*

(iv) $\exists \alpha \in]0, 1[: \delta(f(x), f(y)) \leqslant \alpha\delta(x, y) \ \forall x, y \in X$,

then f has a unique fixed point, x^*, which can be calculated as $\lim_{n\to\infty} f^n(x_0)$ *for any* $x_0 \in X$.

The purpose of the present paper is to give some results about the approximation of the fixed point of f, given by the theorem 1.

1. Sequences of mappings and fixed points. Let X be a topological space and let $f_n, f: X \to X$ be some mappings such that $F_{f_n} \neq \emptyset$ and $F_f = \{x^*\}$. A natural question to ask is the following. Does the convergence of $(f_n)_{n\in N}$ to f imply the convergence of the sequence $(x_n)_{n\in N}$, $x_n \in F_{f_n}$, to the fixed point, x^*, of f?

Some answers to this question are given in [3], [4], [5], [6], [10], [13], [15], [18], [22].

The result which correspond to theorem 1 is the following

THEOREM 2. *Let X be a nonempty set, d and δ two metrics on X and $f_n, f: X \to X$. If*

(a) (X, d), (X, δ) *and f satisfy the hypotheses of theorem* 1,

(b) *the sequence* $(f_n)_{n\in N}$ *converges uniformly on* (X, d) *to* f,

(c) $F_{f_n} \neq \emptyset$,

(d) $\exists c_1 > 0: \delta(x, y) \leq c_1 d(x, y)$, $\forall x, y \in X$,

(e) $cc_1\alpha < 1$,

then for every $x_n \in F_{f_n}$, the sequence $(x_n)_{n\in N}$ converge in (X, d) to the unique fixed point, x^*, of f.

Proof. From the condition (a) and (d) we have

$$d(x_n, x^*) = d(f_n^2(x_n), f^2(x^*)) \leq$$
$$\leq d(f_n^2(x_n), f^2(x_n)) + d(f^2(x_n), f^2(x)) \leq$$
$$\leq d(f_n^2(x_n), f^2(x_n)) + c\delta(f(x_n), f(x)) \leq$$
$$\leq d(f_n^2(x_n), f^2(x_n)) + c\alpha\delta(x_n, x) \leq$$
$$\leq d(f_n^2(x_n), f^2(x_n)) + c\alpha c_1 d(x_n, x)$$

and by (e)

$$d(x_n, x) \leq (1 - cc_1\alpha)^{-1} d(f_n^2(x_n), f^2(x_n)).$$

But

$$d(f_n^2(x_n), f^2(x_n)) \leq$$
$$\leq d(f_n^2(x_n), f(f_n(x_n))) + d(f(f_n(x_n)), f^2(x_n)) \leq$$
$$\leq d(f_n(f_n(x_n)), f(f_n(x_n))) + c\delta(f_n(x_n), f(x_n)) \leq$$
$$\leq d(f_n(f_n(x_n)), f(f_n(x_n))) + cc_1 d(f_n(x_n), f(x_n)).$$

Since $(f_n)_{n\in N}$ converge in (X, d) uniformly to f, hence

$$d(f_n(f_n(x_n)), f(f_n(x_n))) \to 0, \text{ when } n \to \infty,$$

and

$$d(f_n(x_n), f(x_n)) \to 0, \text{ when } n \to \infty,$$

i.e., $(x_n)_{n\in N}$ converge in (X, d) to x^*.

2. Dependence of parameter. Let X and Y be two topological spaces and let: $f: X \times Y \to X$, be a mapping. The induced mappings of f are denoted by f_x, f_y, i.e.:

$$f_x: Y \to X,\ y \to f(x, y),$$
$$f_y: X \to X,\ x \to f(x, y).$$

Let us assume that the mapping f_y has a unique fixed point, x_y^*, for every $y \in Y$. We define the mapping

$$P: Y \to X,\ y \to x_y^*.$$

What are the condition for the continuity of the mapping P? In some particular cases this problem was studied in [10], [12], [13], [14], [15], [18].

Our result for this problem is given in the following theorem.

THEOREM 3. *Let X be a nonempty set, d and δ two metrics on X, and let Y be Hausdorff topological space. If*

(a) *the mapping f_y satisfies the hypotheses of theorem* 1, *where α and c are not depending of y;*

(b) *the mapping f_x is continuous in* (X, d).

(c) $\exists c_1 > 0:\ \delta(x, y) \leq c_1\, d(x, y)$, $\forall x, y \in X$,

(d) $c_1 c\alpha < 1$,

then

$P: Y \to (X, d)$ *is continuous.*

Proof. Let $x_{y_i}^*$, $i = 1, 2$, be the fixed point of f_{y_i}. We have

$$d(x_{y_1}^*, x_{y_2}^*) = d(f_{y_1}^2(x_{y_1}^*), f_{y_2}^2(x_{y_2}^*)) \leq$$
$$\leq d(f_{y_1}^2(x_{y_1}^*), f_{y_1}^2(x_{y_2}^*)) + d(f_{y_1}^2(x_{y_2}^*), f_{y_2}^2(x_{y_2}^*)) \leq$$
$$\leq c\alpha c_1 d(x_{y_1}^*, x_{y_2}^*) + d(f_{y_1}^2(x_{y_2}^*), f_{y_2}^2(x_{y_2}^*))$$

or

$$d(x_{y_1}^*, x_{y_2}^*) \leq (1 - cc_1\alpha)^{-1} d(f_{y_1}^2(x_{y_2}^*), f_{y_2}^2(x_{y_2}^*)).$$

From the condition (a) we have

$$d(x_{y_1}^*, x_{y_2}^*) \to 0, \text{ when } y_1 \to y_2.$$

The proof is complete.

3. The roundoff error in computing the fixed point given by theorem 1. Let $f: X \to X$ be a mapping which satisfies the conditions of theorem 1. Let x^* be the unique fixed point of f. Let $g: X \to X$ be a mapping which approximates the mapping f. More precisely we assume that

$$d(f(x), g(x)) \leq \eta,\ \forall x \in X, \text{ with given } \eta \in R.$$

If $x_0 \in X$, we put

$$x_n = f^n(x_0),\ y_n = g^n(x_0)$$

We propose to give an estimate of $d(x^*, y_n)$. We have

THEOREM 4. *If*

(i) *f satisfies the conditions of the theorem* 1,

(ii) $\exists c_1 > 0: \delta(x, y) \leq c_1 d(x, y)$, $\forall x, y \in X$,

(iii) $cc_1 < 1$.

Then

(a) $d(x_n, x^*) \leqslant \frac{c\alpha^{n-1}}{1-\alpha}\delta(x_0, x_1)$,

(b) $d(y_n, x^*) \leqslant \eta(1 - cc_1)^{-1} + \frac{c\alpha^{n-1}}{1-\alpha}\delta(x_0, x_1)$.

Proof. (a) We have

$$d(x_n, x^*) \leqslant c\delta(x_{n-1}, x^*) \leqslant \frac{c\alpha^{n-1}}{1-\alpha}\delta(x_0, x_1).$$

(b) We have

$$d(y_n, x^*) \leqslant d(x_n, y_n) + d(x_n, x^*)$$

and

$$d(x_n, y_n) \leqslant d(g(y_{n-1}), f(y_{n-1})) + d(f(y_{n-1}), f(x_{n-1})) \leqslant$$
$$\leqslant \eta + cc_1 d(y_{n-1}, x_{n-1}) \leqslant \ldots \leqslant$$
$$\leqslant \eta(1 - (cc_1)^{n-1})(1 - cc_1)^{-1}.$$

4. Example. We take $X = C(\bar\Omega)$, where $\Omega \subset R^n$ is a bounded domaine; $d(x, y) = \max_{t\in\bar\Omega} |x(t)| - y(t)| = \|x - y\|_{C(\bar\Omega)}$, $\delta(x, y) = \left(\int_\Omega (x(t) - y(t))^2 dt\right)^{1/2} = \|x - y\|_{L^2(\Omega)}$. Let $f: C(\bar\Omega) \to C(\bar\Omega)$, be given by

$$f(x)(t) = \int_\Omega K(t, s, x(s))ds$$

where $K \in C(\bar\Omega \times \bar\Omega \times \mathbf{R})$.

If $|K(t, s, u) - K(t, s, v)| \leqslant L(t, s)|u - v|$, $\forall t, s \in \Omega$, $u, v \in R$ we have

$$\alpha = \left(\int_{\Omega\times\Omega} |L(t, s)|^2 dt\ ds\right)^{1/2},$$

$$c = \max_t \left(\int_\Omega (L(t, s)^2 ds\right)^{1/2},$$

$$c_1 = m(\Omega).$$

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Universitatea Babeş–Bolyai
Facultatea de Matematică
Cluj-Napoca