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# ON THE CONTINUITY OF CONVEX MAPPINGS

by

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Let $M$ be a non-empty convex subset of the Euclidean $n$-dimensional linear space $R^n$, and let $R^p$ be the Euclidean $p$-dimensional linear space endowed with the usual (coordinatewise) ordering. It is well known (see for example BRECKNER W. W. [1, pp. 158—159]) that any convex mapping from $M$ into $R^p$ is continuous on the interior of $M$. Therefore it is natural to ask whether this result remains true if the space $R^p$ is replaced by an ordered topological linear space $Y$. An affirmative answer to this question has been given by LOPES PINTO A. J. B. [5, p. 260, Corollary 3.1.2] in the case when $Y$ is assumed to be ordered by a closed normal generating cone. In the present note we shall show that a more general result holds: any convex mapping $f: M \to Y$ is continuous on the interior of $M$ even if the range space $Y$ has only the boundedness property.

## 1. Fundamental definitions

In order to fix the terminology we recall in this section some basic definitions from the theory of ordered topological linear spaces. For detailed information on ordered topological linear spaces we refer the reader to JAMESON G. [3] and to WONG Y.-C. and NG K.-F. [6].

By an *ordered linear space* $Y$ we mean a real linear space $Y$ on which there is defined a binary relation $\leq$ such that for all $x, y, z \in Y$ the following conditions are satisfied:

(i) $x \leq x$;
(ii) $x \leq y$ and $y \leq z$ imply $x \leq z$;
(iii) $x \leq y$ implies $x + z \leq y + z$;
(iv) $x \leq y$ implies $ax \leq ay$ for all real numbers $a > 0$.

The conditions (i) and (ii) express that $\leq$ is an *ordering*, while (iii) and (iv) express the compatibility of the ordering with the linear structure of $Y$.

The *positive wedge* of an ordered linear space $Y$ is the set $Y^+$ of all elements $x \in Y$ such that $o \leq x$, where $o$ denotes the zero-element of $Y$. It is easily seen that $Y^+$ is a wedge*, i.e. a non-empty convex set closed under multiplication by non-negative real numbers.

Conversely, if $Z$ is a wedge in a real linear space $Y$, the binary relation $\leq$ defined by

$$x \leq y \text{ if } y - x \in Z \tag{1.1}$$

verifies conditions (i) — (iv) for all $x, y, z \in Y$, and in consequence makes $Y$ into an ordered linear space whose positive wedge is exactly $Z$. The relation $\leq$ defined by (1.1) is called the *ordering induced by* $Z$.

If $Y$ is an ordered linear space and $x$ and $y$ are elements of $Y$, the set

$$[x, y] = \{z \in Y : x \leq z \text{ and } z \leq y\}$$

is called the *order-interval* between $x$ and $y$. Clearly, $[x, y]$ is non-empty if and only if $x \leq y$.

Let $M$ be a subset of an ordered linear space. $M$ is called *order-bounded* if it is contained in some order-interval. $M$ is said to be *full* (or *order-convex*) if $[x, y] \subseteq M$ for all $x, y \in M$.

An *ordered topological linear space* is defined to be an ordered linear space which is also a real topological linear space. It should be noted that no relation is postulated between topology and ordering except that arising indirectly through their mutual relationship with the linear structure of the space.

An ordered topological linear space is said to be *locally full* (or *locally order-convex*) if it admits a neighbourhood-base at $o$ consisting of full sets. A wedge $Z$ in a real topological linear space $Y$ is said to be *normal* if the ordering induced by $Z$ makes $Y$ into a locally full ordered topological linear space.

We shall say that an ordered topological linear space $Y$ has the *boundedness property* if every order-bounded subset $M$ of $Y$ is bounded, i.e. for every neighbourhood $U$ of $o$ there exists a real number $a > 0$ such that $M \subseteq aU$. Obviously, an ordered topological linear space has the boundedness property if and only if every order-interval is bounded. In particular, every locally full ordered topological linear space has the boundedness property. The converse of this assertion is not true as the following simple example shows.

E x a m p l e. Consider the real linear space $R^2$ equipped with the usual (coordinatewise) ordering and with the topology induced by the semi-norm $\|\cdot\| : R^2 \to R$, defined by

$$\|x\| = |x^1 - x^2| \text{ for all } x = (x^1, x^2) \in R^2.$$

* Some writers use the term cone instead of wedge.



The ordered topological linear space obtained in this way has the boundedness property, but it is not locally full. Indeed, if $x = (x^1, x^2)$ and $y = (y^1, y^2)$ are arbitrary elements of the space such that $x \leq y$, we have

$$\|z\| \leq \|z - x\| + \|x\| \leq z^1 - x^1 + z^2 - x^2 + \|x\| \leq$$
$$\leq y^1 + y^2 - x^1 - x^2 + \|x\|$$

for all $z = (z^1, z^2) \in [x, y]$. Hence the order-interval $[x, y]$ is bounded. Therefore our space has the boundedness property. — Consider now the sequences $x_n = (n, 0)$, $y_n = (n, n)$. They have the properties $o \leq x_n \leq y_n$ and $\lim y_n = o$. Since $(x_n)$ does not converge to $o$, a characteristic property of the locally full ordered topological linear spaces (see WONG Y.-C. and NG K.-F. [6, p. 48, Theorem 5.1]) is not satisfied. Thus the space is not locally full.

## 2. The continuity theorem

Let $M$ be a non-empty convex subset of a real or complex linear space, and let $Y$ be an ordered linear space. A mapping $f : M \to Y$ is said to be *convex* if

$$f(ax + (1 - a)y) \leq af(x) + (1 - a)f(y)$$

for every real number $a \in ]0, 1[$ and every $x, y \in M$. It is an elementary exercise to verify that $f : M \to Y$ is convex if and only if

$$f\left(\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i=1}^{n} a_i f(x_i)$$

for all real numbers $a_1, \ldots, a_n \geq 0$ with $a_1 + \ldots + a_n = 1$ and all elements $x_1, \ldots, x_n \in M$.

For convex mappings defined on convex subsets of Hausdorff topological linear spaces of finite dimensions whose values lie in an ordered topological linear space the following theorem holds:

THEOREM 1. *Let $M$ be a non-empty convex subset of a real or complex Hausdorff topological linear space $X$ of dimension $n \geq 1$, and let $Y$ be an ordered topological linear space having the boundedness property. Then any convex mapping $f : M \to Y$ is continuous on the interior of $M$.*

*Proof.* Let $K$ be the real number field if $X$ is a real linear space, respectively the complex number field if $X$ is a complex linear space. By a well-known result (see for example KÖTHE G. [4, p. 154, (1)]) there exists an isomorphism $A$ between $X$ and the topological linear space $K^n$ in its natural (product) topology. On the set $A(M)$, which obviously is convex, we define the mapping $g$ by

$$g(x) = f(A^{-1}(x)) \text{ for all } x \in A(M).$$

This mapping is convex. We prove that it is continuous on the interior of $A(M)$.

Let $x_0$ be an interior point of $A(M)$, and let $V$ be an arbitrary neighbourhood of the origin of $Y$. Choose, on the one hand, a real number $r_0$ in the open interval $]0, 1[$ such that $x_0 + W \subseteq A(M)$, where $W$ denotes the set of all points $x = (x^1, \ldots, x^n) \in K^n$ with the property

$$|x^1| + \ldots + |x^n| \leq r_0,$$

and, on the other hand, a balanced neighbourhood $V_{m+2}$ of the origin of $Y$ such that

$$\underbrace{V_{m+2} + \ldots + V_{m+2}}_{m+2 \text{ terms}} \subseteq V, \tag{2.1}$$

where $m = n$ if $X$ is a real linear space and $m = 2n$ if $X$ is a complex linear space. Since $Y$ has the boundedness property and the set ext $W$ of all extremal points of $W$ is finite, there exists a real number $r \in ]0, r_0]$ such that the following two properties hold:

$$r\left[o, \sum_{i=1}^{m+1} (g(x_0 + x_i) - 2g(x_0) + g(x_0 - x_i))\right] \subseteq V_{m+2} \tag{2.2}$$

for all $x_1, \ldots, x_{m+1} \in \text{ext } W$;

$$r(g(x_0) - g(x_0 - x)) \in V_{m+2} \text{ for all } x \in \text{ext } W. \tag{2.3}$$

Put now $U = x_0 + rW$. This set is a neighbourhood of $x_0$ contained in $A(M)$. We claim that

$$g(x) - g(x_0) \in V \text{ for all } x \in U. \tag{2.4}$$

Indeed, if $x$ is an arbitrary element of $U$, it must be of the form $x = x_0 + ry$ for a suitable element $y \in W$. According to the convexity of $g$, the equality

$$x = r(x_0 + y) + (1 - r)x_0$$

implies

$$g(x) \leq rg(x_0 + y) + (1 - r)g(x_0). \tag{2.5}$$

Analogously,

$$x_0 = \frac{r}{1+r}(x_0 - y) + \frac{1}{1+r}x$$

implies

$$g(x_0) \leq \frac{r}{1+r}g(x_0 - y) + \frac{1}{1+r}g(x). \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$r(g(x_0) - g(x_0 - y)) \leq g(x) - g(x_0) \leq r(g(x_0 + y) - g(x_0)). \tag{2.7}$$



By a well-known result (see for example HOLMES R. B. [2, p. 82, Lemma b]) $y$ is a convex combination of at most $m + 1$ points of ext $W$, i.e. there exist $m + 1$ real numbers $a_1, \ldots, a_{m+1} \geq 0$ with $a_1 + \ldots + a_{m+1} = 1$ and $m + 1$ points $x_1, \ldots, x_{m+1}$ of ext $W$ such that

$$y = a_1x_1 + \ldots + a_{m+1}x_{m+1}.$$

Hence we have

$$g(x_0 - y) = g\left(\sum_{i=1}^{m+1} a_i(x_0 - x_i)\right) \leq \sum_{i=1}^{m+1} a_ig(x_0 - x_i),$$

$$g(x_0 + y) = g\left(\sum_{i=1}^{m+1} a_i(x_0 + x_i)\right) \leq \sum_{i=1}^{m+1} a_ig(x_0 + x_i).$$

Together with (2.7) these inequalities imply

$$r\sum_{i=1}^{m+1} a_i(g(x_0) - g(x_0 - x_i)) \leq g(x) - g(x_0) \leq$$

$$\leq r\sum_{i=1}^{m+1} a_i(g(x_0 + x_i) - g(x_0)),$$

and so we obtain

$$o \leq g(x) - g(x_0) - r\sum_{i=1}^{m+1} a_i(g(x_0) - g(x_0 - x_i)) \leq \tag{2.8}$$

$$\leq r\sum_{i=1}^{m+1} a_i(g(x_0 + x_i) - 2g(x_0) + g(x_0 - x_i)).$$

Since $a_i \leq 1$ and $o \leq g(x_0 + x_i) - 2g(x_0) + g(x_0 - x_i)$ for all $i = 1, \ldots, m + 1$, we get from (2.8)

$$g(x) - g(x_0) - r\sum_{i=1}^{m+1} a_i(g(x_0) - g(x_0 - x_i)) \in$$

$$\in r\left[o, \sum_{i=1}^{m+1} (g(x_0 + x_i) - 2g(x_0) + g(x_0 - x_i))\right].$$

Taking (2.2) into account, it follows then that

$$g(x) - g(x_0) - r\sum_{i=1}^{m+1} a_i(g(x_0) - g(x_0 - x_i)) \in V_{m+2},$$

whence

$$g(x) - g(x_0) \in V_{m+2} + r\sum_{i=1}^{m+1} a_i(g(x_0) - g(x_0 - x_i)).$$

By (2.3) and (2.1) we have then

$$g(x) - g(x_0) \in \underbrace{V_{m+2} + \cdots + V_{m+2}}_{m+2 \text{ terms}} \subseteq V.$$

So (2.4) is proved.

Since $V$ was an arbitrary neighbourhood of the origin of $Y$, (2.4) shows that $g$ is continuous at $x_0$. Hence $g$ is continuous on the whole interior of $A(M)$. It follows then that $f$ is continuous on the interior of $M$. ∎

*Remark* 1. In theorem 1 the hypothesis that $X$ is a Hausdorff space cannot be dropped as shown by the following example. Let $X$ be the topological linear space considered in the example given in the first section. The mapping $f: X \to R$ defined by $f(x) = x^1$ for all $x = (x^1, x^2) \in X$ is linear, hence convex, but not continuous.

*Remark* 2. In theorem 1 the hypothesis that $Y$ has the boundedness property cannot be dropped either. To show this, set $X = R$ and $Y = C^2[1/2, 1]$, where $C^2[1/2, 1]$ denotes the ordered normed linear space of all twice continuously differentiable functions $y$: $[1/2, 1] \to R$ with the norm

$$\|y\| = \sum_{i=0}^{2} \max \{|y^{(i)}(t)| : t \in [1/2, 1]\}$$

and with the pointwise ordering. $Y$ does not have the boundedness property, since the order-interval between the origin and the function $y$: $[1/2, 1] \to R$, defined by $y(t) = 1$ for all $t \in [1/2, 1]$, is not bounded. Put $M = [-1, 1]$. The mapping $f: M \to Y$, defined by

$$(f(x))(t) = \begin{cases} x^4 t^{1/x^2} & \text{for } x \in M \setminus \{0\} \\ 0 & \text{for } x = 0 \end{cases}$$

for all $t \in [1/2, 1]$, is convex, but it is not continuous at the point $x_0 = 0$. The latter assertion follows from the inequality

$$\|f(x) - f(x_0)\| \geq \left| \frac{d^2 f(x)}{dt^2}(1) - \frac{d^2 f(x_0)}{dt^2}(1) \right| = 1 - x^2 \geq \frac{3}{4},$$

valid for all $x \in [-1/2, 1/2] \setminus \{0\}$.

COROLLARY 2. *Let $M$ be a non-empty convex subset of a real or complex Hausdorff topological linear space $X$ of dimension $n \geq 1$, and let $Y$ be a locally full ordered topological linear space. Then any convex mapping $f: M \to Y$ is continuous on the interior of $M$.*

COROLLARY 3. *Let $X$ and $Y$ be real or complex topological linear spaces over the same field with $X$ Hausdorff and finite-dimensional. Then any linear mapping $f: X \to Y$ is continuous.*



*Proof.* Let $Y_R$ be the real restriction of $Y$. The ordering induced by $\{o\}$ in $Y_R$ makes $Y_R$ into an ordered topological linear space which is locally full. The assertion follows then by applying corollary 2 to the mapping $f: X \to Y_R$. ∎

*Remark* 3. In the proof of theorem 1 we made use of the fact that any real or complex Hausdorff topological linear space of dimension $n \geq 1$ is isomorphic to the topological linear space $K^n$ in its natural (product) topology. However, this result can be deduced by applying corollary 3. Hence theorem 1, corollary 2, corollary 3 and the above-mentioned result on the isomorphism of the finite-dimensional topological linear spaces are equivalent.

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