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ON THE ZEROS OF ORTHOGONAL POLYNOMIALS WITH RESPECT TO MEASURES WITH NONCOMPACT SUPPORT*

by

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1. Introduction.

Let $0 \leqslant v(x) \in L$ be a weight function defined on $-\infty < x < \infty$ for wich all moments

$$(1) \qquad \mu_n(w) = \int_{-\infty}^{\infty} x^n w(x) dx \qquad (n = 0, 1, \ldots)$$

are finite. Then there exist a uniquely defined sequence $\{p_n(w\,;\, x)\}$ of polynomials orthonormal with respect to the measure $w\,dx$ so that the degree of $p_n(w)$ is precisely n and the coefficient $\gamma_n(w)$ of x^n is positive. The zeros of $p_n(w\,;\, x)$ are all real and simple (see e.g. [3]) ; we denote them in decreasing order by $x_{kn}(w)$ $(k = 1, 2, \ldots, n)$. In the present survay of results and problems we are interested in the case when the support of $w\,dx$ is not compact. We are going to reveiw our investigations in the last five years and complete it with some more recent observations. The only wellknown cases are if the weight is

$$(2) \qquad v_\alpha(x) = \begin{cases} x^\alpha e^{-x} & (x > 0) \\ 0 & (x \leqslant 0) \end{cases} \qquad (\alpha > -1)$$

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and

(3) $$w_\beta(x) = |x|^\beta e^{-x^2} \quad (\infty < x < \infty;\ \beta > -1).$$

The orthogonal polynomials $p_n(v_\alpha;\ x)$ are equal, appart from constant factors, to the classical Laguerre polynomials wich have a welldeveloped theory (see e.g. G. SZEGŐ [12]).

The polynomials $p_n(w_\alpha;\ x)$ — the socalled Markov-Sonin polynomials — can be reduced by the formulae

(4) $$p_{2n}(w_\beta;\ x) = p_n(v_{\frac{\beta-1}{2}};\ x^2), \quad x_{kn}(v_{\frac{\beta-1}{2}}) = [x_{k,2n}(w_\beta)]^2$$

and

(5) $$p_{2n+1}(w_\beta;\ x) = x\, p_n(v_{\frac{\beta+1}{2}};\ x^2), \quad x_{kn}(v_{\frac{\beta+1}{2}}) = [x_{k,2n+1}(w_\beta)]^2.$$

For $\beta = 0$, $p_n(w_0;\ x)$ is a constant multiple of the Hermite orthogonal polynomial $H_n(x)$. Having the transformations (4) and (5) in mind also theory of the polynomials $p_n(w_\beta;\ x)$ can be considered as settled but not much was known for cases if the weights are not known by explicit analytic formulae.

2. „Near to Laguerre" and „near to Hermite" polynomials

In our first investigation we treated the class Λ of „near to Laguerre" weights. A weight w belongs to Λ iff $w(x) = 0$ for $x \leq 0$ and there exist two positive numbers a, r so that for every pair $0 < x_1 \leq x_2$ we have

(6) $$e^{x_1} w(x_1) \leq a e^{x_2} w(x_2)$$

and

(7) $$x_1^{-r} e^{x_1} w(x_1) \geq x_2^{-r} e^{x_2} w(x_2).$$

We proved in our paper [4] (see also [5]) that

(8) $$x_{1n}(w) \leq 4n + O\left(n^{\frac{1}{3}}\right) \quad (w \in \Lambda)$$

and

(9) $$c_1(w) n^{-1} \leq x_{nn}(w) \leq c_2(w) n^{-1} \qquad (w \in \Lambda).$$

Let further $x_{nn}(w) < \xi < x_{1n}(w)$ and k an integer for wich

$$x_{kn}(w) \geq \xi \geq x_{k+1,n}(w),$$

then we have

(10) $$c_3(w)\varphi_n(\xi) \leq x_{kn}(w) - x_{k+1,n}(w) \leq c_4(w)\varphi_n(\xi) \quad (w \in \Lambda),$$

where

(11) $$\varphi_n(\xi) = \frac{\xi^{\frac{1}{2}} + n^{-\frac{1}{2}}}{|4n - \xi|^{\frac{1}{2}} + n^{\frac{1}{6}}} \, (\xi \geq 0).$$

As an extension of this result we define the class $\acute{X}$ of „near to Hermite" weights by the following properties: $w \in \acute{X}$ iff w is even and $x^{-\frac{1}{2}} w\left(x^{\frac{1}{2}}\right) \in \Lambda$. Seting

(12) $$v(x) = x^{-\frac{1}{2}} w\left(x^{\frac{1}{2}}\right), \quad V(x) = x^{\frac{1}{2}} w\left(x^{\frac{1}{2}}\right) \quad (x > 0)$$

we have

(13) $$p_{2n}(w;\ x) = p_n(v;\ x^2), \quad p_{2n+1}(w;\ x) = x\, p_n(V;\ x)$$

(see e.g. [3], problems I.13 and I.14).

By this transformation formulae we obtain

THEOREM 1. *We have for every* $w \in \acute{X}$

(14) $$X_{1n}(w) = -X_{nn}(w) \leq \sqrt{2n} + O\left(n^{-\frac{1}{6}}\right)$$

and for every $|\xi| < x_{1n}(w)$

(15) $$x_{k+1,n}(w) \leq \xi \leq x_{kn}(w)$$

implies

(16) $$C_5(w)\Phi_n(\xi) \leq X_{kn}(w) - X_{k+1,n}(w) \leq C_6(w)\Phi_n(\xi),$$

where

(17) $$\Phi_n(\xi) = \left(|2n - \xi| + n^{-\frac{1}{3}}\right)^{-\frac{1}{2}}$$

PROBLEM A (Unsolved). *Let* $w \in \acute{X}$; *when does the limit*

(18) $$\lim_{n\to\infty} \frac{X_{kn}(w) - X_{k+1,n}(w)}{\Phi_n(\xi)}$$

exist for every fixed ξ?

Problem A is a seemingly accessible special case of a more general unsolved problem posed at the end of our book [3]. The only case wich can be settled directly is $w \equiv w_\beta$ applying the known properties of the zeros of Laguerre polynomials we obtain by (4) and (5) that the limit (18) exists and is equal to π for every fixed value of ξ.

3. Estimation of the greatest zero

Let

(19) $$w_Q(x) = e^{-Q(x)},$$

where $Q(x)$ is an even differentiable function for wich $x^\delta Q'(x)$ $(x > 0)$ is an increasing function for some $\rho < 1$.

Defining q_n as the unique positive solution of the equation

(20) $$q_n Q'(q_n) = n$$

we proved in [7] that

(21) $$c_7(w_Q) q_n \leqq x_{1n}(w_Q) \leqq c_8(w_Q) q_n.$$

Special cases of (21) where proved earlyer in [6] and by G.P. NÉVAI [11].

The estimate (21) can be combined with the following

L e m m a 1 (see [8]). *If w_1 and w_2 are even weights and A, $B > 0$,*

(22) $$A w_1(x) \leqq w_2(x) \leqq B w_1(x) \quad (-\infty < x < \infty),$$

then we have

(23) $$\frac{A}{B} x_{1n}(w_1) \leqq x_{1n}(w_{2n}) \leqq \frac{B}{A} x_{1n}(w_1).$$

If w_2 is not necessarly even (23) *holds only after replacing $x_{1n}(w_2)$ by $X_n(w_2) = \max\{x_{1n}(w_2) - x_{nn}(w_2)\}$.*

The proof of (21) is based on

L e m m a 2 (see [6]). *We have for every even weight w*

(24) $$\Gamma_{n-1}(w) \leqq x_{1n}(w) \leqq 2\Gamma_{n-1}(w),$$

where

(25) $$\Gamma_\nu(w) = \max_{k \leqslant \nu} \frac{\gamma_{k-1}(w)}{\gamma_k(w)}.$$

In (24) neither the lower estimate can be improved by a factor greater then one nor is the upper estimate valid in general if [2] is replaced by a smaller factor (see [6]). But we found recently that if γ_{k-1}/γ_k varies in a regular manner then the upper estimate in (24) is assymptotic:

L e m m a 3 (see [9]). *Let w by even, and let the sequence $d_k(w) = \frac{\gamma_{k-1}(w)}{\gamma_k(w)}$ satisfy*

(26) $$\lim_{k\to\infty} \frac{d_{k+1}(w)}{d_k(w)} = 1,$$

then we have

(27) $$\lim_{n\to\infty} \frac{X_{1n}(w)}{(\Gamma_{n-1}w)} = 2.$$

Lemma 3 can be combined with the following

PROBLEM B (conjectured in [9]). *Let $w_{\beta\rho}(x) = |x|^\beta e^{-|x|^\rho}$ $(-\infty < x < \infty;\ \beta > -1,\ \rho > 0)$, then*

(28) $$\lim_{n\to\infty} n^{-\frac{1}{\rho}} d_n(w_{\beta\rho}) = \left[\frac{\Gamma(\rho+1))}{\Gamma\left(\frac{\rho}{2}\right)\Gamma\left(\frac{\rho}{2}+1\right)}\right]^{-\frac{1}{\rho}}.$$

(28) was proved by us in [9] for $\rho = 2$, 4 and 6 and arbitrary $\beta > -1$. Whenever (28) is true (i.e. in particular for $\rho = 2$, 4 and 6) we infer from Lemma 3 that

(29) $$\lim_{n\to\infty} n^{-\frac{1}{\rho}} x_{1n}(w_{\beta\rho}) = 2\left[\frac{\Gamma(\rho+1)}{\Gamma\left(\frac{\rho}{2}\right)\Gamma\left(\frac{\rho}{2}+1\right)}\right]^{-\frac{1}{\rho}}.$$

In the cases that $-\log w(x)$ increases substantially more rapidly then a power of x the assymptotic of $x_{1n}(w)$ was setled by the following

THEOREM 2 (P. ERDŐS [1]). *Let $w_R(x) = \exp\{-R(x)\}$, let for every $\varepsilon > 0$*

(30) $$R(y) > 2R(x) \quad (|y| > (1+\varepsilon)|x| > c_9(R;\ \varepsilon))$$

and let

(31) $$w_R(\xi_n) = 2^{-n} \quad (\xi_n > 0),$$

then we have

(32) $$\lim_{n\to\infty} \frac{X_{1n}(w_R)}{\xi_n} = 1$$

4. The distance of consecutive zero

Let $\xi \in [x_{nn}(w),\ x_{1n}(w)]$,

(33) $$x_{kn}(w) \geqq \xi \geqq x_{k+1,n}(w)$$

and

(34) $$\Delta_n(w;\ \xi) = x_{kn}(w) - x_{k+1,n}(w).$$

THEOREM 3. *Let* $w_Q(x) = \exp\{-Q(x)\}$ *where* Q *is an even differentiable convex function for which* $1 < \delta < \frac{Q'(2x)}{Q'(x)} < 0$ *and let the sequence* $\{q_n\}$ *be defined by* (20) *then*

$$\Delta_n(w_Q\,;\ \xi) \leqq C_{10}(w_Q)\frac{q_n}{n} \quad (|\xi| \leqq C_{11}(w_Q)q_n). \tag{35}$$

We hope to return to the proof of Theorem 3 elsewhere.

PROBLEM C. *Is* (35) *valid with* $C_{10}(w_Q)$ *replaced by* $C_{10}(\delta, w_Q)$ *for every* $\delta > 0$ *and* $|\xi| \leqq (1-\delta)x_{1n}(w_Q)$?

The answer to Problem C is affirmative for the weights defined by (3); no other case seems to be setled.

PROBLEM D. *Do the conditions of Theorem* 3 *imply that*

$$\Delta_n(w_Q\,;\ \xi) \geqq C_{12}(w_Q)\frac{q_n}{n}\ ? \tag{36}$$

We know that the answer to Problem D is ,,yes'' for a rather extented classe of weights, in particular for the weights $w_{0\rho}(x)$ (defined in Problem *B*) whenever $\rho > 1$. We conjecture that the answer to Problem *D* is affirmative in general.

Special cases of Theorem 3 as well as of Problem *D* where treated by G. P. NÉVAI [10].

5. The distribution function of the zeros

For a $0 < \eta < 1$ we define $\nu_n(w\,;\ \eta)$ as the number of the zeros of $p_n(w)$ wich are greater then $x_{1n}(w) - \eta[x_{1n}(w) - x_{nn}(w)]$. The distribution function of the zeros is defined as

$$\lim_{n\to\infty}\frac{1}{n}\,\nu_n(w\,;\ \eta) = f(\eta). \tag{37}$$

In our book [3] we called the attention to the *unsolved problem* to give *conditions for the existence of* $f(\eta)$.

It is welknown that the zeros of the classical Jacobi polynomials have the distribution function

$$f_0(\eta) = \frac{\pi}{2} - \arcsin\ \eta. \tag{38}$$

This can be interpreted that the numbers $\delta_{kn} = \arcsin\ x_{kn}$ are equidistributed in H. Wely's sense. P. ERDŐS [1] discovered that also the zeros of $p_n(w_R)$ have the distribution function (38). Here w_R ia a weight wich satisfies the conditions of Theorem 2. P. ERDŐS [1] also remarked that the weights $w_{0\rho}$ (see [28]) can not have the distribution function (38) for any $\rho > 0$. A proof of this fact is published in our joint paper [2].

Applying the Plancherel-Rotach assymptotic formula (see e.g. G. SZEGŐ [12]) it is easy to see that the Hermite orthogonal polynomials have the distribution $f_1(\eta)$ defined by

$$f_1(\cos\Phi) = \frac{1}{2\pi}(2\Phi - \sin 2\Phi). \tag{39}$$

The Markov-Sonin polynomials $p_n(w_\beta)$ have the same distribution function (39).

PROBLEM E. *Characterize the weights* w *dor wich the distribution function of* $p_n(w)$ *is* (39).

PROBLEM F. *Determine the distribution function of the roots of* $p_n(w_{0\rho})$ *where* $w_{0\rho}(x) = \exp\{-|x|^\rho\}$ *at least for some* $0 < \rho \neq 2$.

Let us note that not even the existence of the distribution function for the cases $0 < \rho \neq 2$ is proved.

REFERENCES

[1] Erdös, P., *On the distribution of roots of orthogonal polynomials.* Proceedings of the Conference on Constructive Theory of Functions. (Budapest 1969). Edited by G. Alexits and S. B. Steckin. Akadémiai Kiadó, Budapest 1972.

[2] Erdős, P., Freud, G., *On orthogonal polynomials with regularly distributed zeros.* Proceedings of the London Mathem. Soc. (in print).

[3] Freud, G., *Orthogonale Polynome.* Birkhäuser Verl. Basel 1969. English translation by I. Földes: Orthogonal Polynomials, Pergamon Press, New York-Toronto-London, 1971.

[4] Freud, G., (Фреуд Г.), Об одном классе ортогоналных многочленов. Матем. Заметки 9 (97), 5 — 522

[5] Freud, G., *On a class of orthogonal polynomials.* Proceedings of the International Conference on Constructive Function Theory (Varna 1970), Sofia, 1972.

[6] Freud, G., *On the greatest zero of on orthogonal polynomial,* I. Acta Sci. Math. (Szeged) 91—97.

[7] Freud, G., *On the greatest zero on orthogonal polynomial,* II. Acta Sci. Math. (Szeged), (in print).

[8] Freud, G., *On estimations of the greatest zero of an orthogonal polynomial.* Acta Math. Sci. Hung. (in print.).

[9] Freud, G., *On the coefficients in the recursion formulae of orthogonal polynomials.* Studia Sci. Math. Hung. (in print).

[10] Névai, G. P., Некоторые свойства многочленов ортонормальных с весом $(1+x^{2k})^\alpha e^{-x^{2k}}$ и их примеиения в теории приближения Доклады Ак. Наук ССР.

[11] Névai, G. P., Многочлены, ортонормальные на вещественной оси с весом. $x^\alpha e^{-|x|^\beta}$ Acta Math. Ac. Sci. Hung. 24, 335—342 (1973).

[12] Szegő, G., *Orthogonal polynomials.,* Amer. Math. Soc. Coll. Publ., vol. XXIII, 2, revised edition, 1959.

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