MATHEMATICA–REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 6, N° 2, 1977, pp. 139–143

LINEAR POSITIVE OPERATORS ON THE SPACE AC_m

by

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The aim of this paper is to investigate some convergence properties for sequences of linear operators L_n, $n = 1, 2, \ldots$, which preserve a certain cone $K_m = K_m[a, b]$, that is $L_n(K_m) \subseteq K_m$. In the following we shall consider that $K_m[a, b]$ is the cone of all functions from $C[a, b]$ which are convex of the order m on $[a, b]$, $m = 0, 1, \ldots$.

Definition 1. *A function f: $[a, b] \to R$ is called absolutely continuous of m^{th} order, if for every partition*

$$a \leqslant x_0^0 < x_1^0 < \ldots < x_{m+1}^0 \leqslant x_0^1 < x_1^1 < \ldots < x_{m+1}^1 \leqslant \ldots$$

$$\ldots \leqslant x_0^\nu < x_1^\nu < \ldots < x_{m+1}^\nu \leqslant b$$

with

$$\sum_{k=0}^{\nu} (x_{m+1}^k - x_0^k) < \delta(\varepsilon)$$

one has

$$\sum_{k=0}^{\nu} |\, [x_0^k, x_1^k \ldots, x_{m+1}^k ; f]|(x_{m+1}^k - x_0^k) < \varepsilon$$

The space of all functions $f:[a, b] \to R$ which are absolutely continuous of m^{th} order is denoted by $AC_m = AC_m\ [a, b]$.

In [1] it is shown that a function $f:[a, b] \to R$ belongs to the space $AC_m[a, b]$ if and only if f has on $[a, b]$ a derivative of m^{th} order which is absolutely continuous on this interval. In [2] it is proved that $AC_m\ [a, b]$ is a linear subspace of the space $BV_m\ [a, b]$ of all functions with bounded variation of the order m.

On the space $AC_m[a, b]$ let us define the functional $\|\cdot\|_m$ by

(1) $$\|f\|_m = |f(a)| + |f'(a)| + \ldots + |f^{(m)}(a)| + V_m(a, b; f),$$

where $V_m(a, b; f)$ is the m^{th} total variation of the function f.

It may be proved (see [1] and [4]) that $\|\cdot\|_m$ is a norm on the space $BV_m[a, b]$. Moreover, the linear normed space $BV_m[a, b]$ becomes a Banach space.

Definition 2. *An operator $L: AC_m[a, b] \to AC_m[a, b]$ is called m-positive iff $L(\overline{K}_m) \subseteq \overline{K}_m$, where $\overline{K}_m = AC_m[a, b] \cap K_m[a, b]$.*

THEOREM 1. *An operator $L: AC_m[a, b] \to AC_m[a, b]$ is m-positive iff for every $f \in AC_m[a, b]$ with $f^{(m+1)}(x) \geq 0$ a.e. $x \in [a, b]$ it follows that $(Lf)^{(m+1)}(x) \geq 0$ almost everywhere on $[a, b]$.*

Proof. Let us consider that L is a m-positive operator. It is known that a function $f \in \overline{K}_m$ if and only if the derivative $f^{(m+1)}$ exists almost everywhere on $[a, b]$, and $f^{(m+1)} \geq 0$. Taking into account that $Lf \in \overline{K}_m$, it follows $(Lf)^{(m+1)} \geq 0$ almost everywhere on $[a, b]$.

The converse statement may be proved in a similar way.

The main result of this paper is the following:

THEOREM 2. *Let $L_n: AC_m[a, b] \to AC_m[a, b]$, $n = 1, 2, \ldots$, be a sequence of linear m-positive operators which verify:*

a) *there exists a positive number M such that for $n = 1, 2, \ldots$*

$$\|L_n\|_m \leq M;$$

b) *for every $f \in AC_m[a, b]$*

(2) $$\lim_{n\to\infty} |(L_n f)^{(k)}(a) - f^{(k)}(a)| = 0, \quad k = 0, 1, 2, \ldots, n_0.$$

If $e_k(t) = t^k$, $k = 0, 1, \ldots$, $t \in [a, b]$, and

(3) $$\lim_{n\to\infty} \|L_n e_k - e_k\|_m = 0$$

for $k = 0, 1, \ldots, m+3$, then for every $f \in AC_m[a, b]$

(4) $$\lim_{n\to\infty} \|L_n f - f\|_m = 0.$$

Proof. From (1) and (2) it follows that is sufficient to prove

(5) $$\lim_{n\to\infty} V_m(a, b; (L_n f - f)) = 0.$$

If $g \in AC_m[a, b]$ then the total variation of m^{th} order is given by

$$V(a, b; g) = \int_b^a |g^{(m+1)}(t)|\,dt.$$

Therefore we must show that the equality

(6) $$\lim_{n\to\infty} \|(L_n f)^{(m+1)} - f^{(m+1)}\|_{L^1[a,b]} = 0$$

is valid. Our hypothesis enables us to assert that

$$\lim_{n\to\infty} \|(L_n e_k)^{(m+1)} - e_k^{(m+1)}\|_{L^1[a,b]} = 0,$$

for $k = 0, 1, \ldots, m+3$, as well as that the sequences $((L_n f)^{(m+1)})_{n=1,2,\ldots}$ and $((L_n e_k)^{(m+1)})_{n=1,2,\ldots}$, $k = 0, 1, \ldots, m+3$, are uniformly bounded. In conclusion, if we show that (6) is valid for every $f \in C^{(m+1)}[a, b]$ than, by means of the Banach-Steinhaus theorem, our proof will be complete.

Let $f \in C^{(m+1)}[a, b]$; i.e. for any $\varepsilon > 0$ there is a positive number $\delta = \delta(\varepsilon)$ so that for every $x, t \in [a, b]$, $|x - t| < \delta$, one has

$$|f^{(m+1)}(x) - f^{(m+1)}(t)| < \varepsilon.$$

If $g(t, x) = (t - x)^{m+3}$, $x, t \in [a, b]$, then

(7) $$g_t^{(m+1)}(x, t) = \frac{\partial g^{m+1}(x, t)}{\partial t^{m+1}} = C(t - x)^2$$

with $C > 0$. From (7) one concludes that the inequality

$$|f^{(m+3)}(x) - f^{(m+1)}(t)| < \varepsilon + C(t - x)^2$$

is verified for any x, t from $[a, b]$ with $|x - t| < \delta$.

If we put $\varphi = \frac{1}{(m+1)!} e_{m+1}$, the above inequality may be written as

$$-\varepsilon\varphi^{(m+1)}(t) - Cg^{(m+1)}(x, t) < f^{(m+1)}(x)\varphi^{(m+1)}(t) - f^{(m+1)}(t) <$$
$$< \varepsilon\varphi^{(m+1)}(t) + Cg^{(m+1)}(x, t),$$

which implies

$$-\varepsilon(L_n\varphi)^{(m+1)} - C(L_n g)^{(m+1)} < f^{(m+1)}(x)(L_n\varphi)^{(m+1)} - (L_n f)^{(m+1)} <$$
$$< \varepsilon(L_n\varphi)^{(m+1)} + C(L_n g)^{(m+1)},$$

$n = 1, 2, \ldots$. Therefore

$$|f^{(m+1)}(x)(L_n\varphi)^{(m+1)} - (L_n f)^{(m+1)}| < \varepsilon\,(L_n\varphi)^{(m+1)} + C(L_n g)^{(m+1)},$$

for $n = 1, 2, \ldots$; $t, x \in [a, b]$ and $|t - x| < \delta$.

Now, for $x = t$ one obtains

(8) $$|(L_n f)^{(m+1)}(x) - f^{(m+1)}(x)| \leqslant$$
$$|(L_n f)^{(m+1)}(x) - f^{(m+1)}(x)(L_n\varphi)^{(m+1)}(x)| +$$

$$+ |f^{(m+1)}(x)(L_n\varphi)^{(m+1)}(x) - f^{(m+1)}(x)\varphi^{(m+1)}(x)| <$$

$$< \varepsilon(L_n\varphi)^{(m+1)}(x) + C(L_n g)^{(m+1)}(x) + \sup_{x\in[a,b]} |f^{(m+1)}(x)|\cdot|(L_n\varphi)^{(m+1)}(x) - \varphi^{(m+1)}(x)|.$$

By using the hypothesis quoted in this theorem it follows that

(I) $\lim_{n\to\infty} |(L_n\varphi)^{(m+1)}(x) - \varphi^{(m+1)}(x)| = 0$ a.e. $x \in [a, b]$;

(II) $\lim_{n\to\infty} |(L_n\varphi)^{(m+1)}(x)| = 1$ a.e. $x \in [a, b]$;

(III) $f^{(m+1)}$ attains its extreme values on $[a, b]$.

On the other hand $L_n g \in C[a, b]$ and

$$(L_n g) = \sum_{k=0}^{m+1} \binom{m+3}{k} (-1)^k (L_n e_{m+3-k}),\ t,\ x \in [a, b].$$

Choosing $t = x$ we find

$$(L_n g)^{(m+1)}(x) = \sum_{k=0}^{m+3} \binom{m+3}{k} (-1)^k e_k(x) (L_n e_{m+3-k})^{(m+1)})(x),\quad x \in [a, b],$$

which furnishes

$$\lim_{n\to\infty} (L_n g)^{(m+1)} = 0.$$

In conclusion, the right-term in (8) converges to zero and the theorem is proved.

In the particular case $m = 0$, a similar result was obtained by S. STADLER [5].

Examples of m-positive linear operators are: the S. N. Bernstein. operators, [3], and the operators $I_n : CP[a, b] \to CP[a, b]$, defined as

$$(I_n f)(x) = \int_a^b N_n(t) f(x + t)dt.$$

Here, $CP[a, b]$ denotes the subspace of $C[a, b]$ which includes the periodic functions whose periode is $b - a$. Likewise, $N_n \in C[a, b]$ and $N_n \geq 0$ on $[a, b]$.

For these operators we observe that

$$[x_1, x_2, \ldots, x_{m+2};\ I_n f] = \int_a^b N_n(t) [x_1, x_2, \ldots, x_{m+2};\ f(x + t)]\ dt$$

holds for every distinct points $x_1, x_2, \ldots, x_{m+2}$ from $[a, b]$, which enables us to assert that they are m-positive.

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Received 7. VI. 1977.