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# METHOD OF CHEBYSHEV CENTERS FOR BEST LINEAR ONE — SIDED $L_p$ — APPROXIMATION

by

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## 1. Indroduction

Let $A = (a_{ij})$ be a real $m \times n$ matrix (with rank$A = n$) and $b$ a real $m$-dimensional column vector. We design by $a^{i\cdot} = (a_{i1}, a_{i2}, \ldots, a_{in})$, $a^{\cdot j} = (a_{1j}, a_{2j}, \ldots, a_{mj})^T$ row and column vectors of $A$ respectively. The optimization problem:

(1) $$\min \{f(x) | Ax \leq b\},$$

where

(2) $$f(x) = \left( \sum_{i=1}^{m} (b_i - a^{i\cdot} x)^p \right)^{1/p}$$

is called best linear one-sided $L_p$-approximation.

Such problems occur frecvently in analysis as it was indicated by R. BOJANIC and R. DE VORE [1]. One of such a problem is, for example, discretizated one-sided continuous $L_p$-approximation of a given function by generalized polynomials (see for instance [1, 2]).

As it is known, in the case $p = 1$ and $p = \infty$, the problem (1) is equivalent to a linear programming problem. When $1 < p < \infty$, then (1) is a convex programming problem, with linear inequality constraints. There are several methods which can be used to solve such a problem. In 1973, G. A. WATSON [6] gave some numerical results concerning the efficiency of the gradient projection method of J. B. ROSEN [5] and the reduced gradient method of R. WOLFE [7] in solving problem (1). Both these methods involve one-dimensional minimization problem in order to find the length of the step at each iteration. In order to avoid this, usually a linear or quadratic interpolation is used.

In this paper we propose the application of the method of Chebyshev centers of A. JU. LEVIN [3] and S. I. ZUHOVICKII — M. E. PRIMAK [8] to solve (1). This is a special cutting plane method which involves at each iteration to solve a linear programming problem. Beeing an interior points method, no step-length calculation is needed.

Before proceeding, we note that the problem (1) may be simplified by replacing the objective function $f$ by its $p$-th power ; clearly, the same solution $x^*$ solves both problems. Thus, we will deal with the problem :

(3) $$\min\{F(x) | Ax \leq b\},$$

where

(4) $$F(x) = \sum_{i=1}^{m} (b_i - a^{i\cdot}x)^p, \qquad p > 1.$$

## 2. Method of Chebyshev centers

Consider a convex programming problem :

(5) $$\min\{f(x) | Ax \leq b\}$$

where $f$ is assumed to be convex and possessing a subgradient at each point of the set

$$\Omega_0 = \{x \in R^n | Ax \leq b\},$$

with $\text{int}\Omega_0 \neq \Phi$. Let us denote by $\partial f(x)$ a subgradient of the function $f$ in the point $x$, i.e.

$$\partial f(x)(y - x) \leq f(y) - f(x), \ \forall y \in R^n.$$

The algorithm of Chebyshev centers for the problem (5) consists in the following steps.

First a feasible solution of the problem (5) is selected, assume $x^1 \in \Omega_0$. Then Chebyshev center $x^2 \in \Omega_0$ is determined of the system of inequalities :

$$Ax \leq b$$
$$a^{m+1\cdot}x \leq b^{m+1},$$

where

$$a^{m+1\cdot} = \partial f(x^1), \ b_{m+1} = \partial f(x^1)x^1,$$

i.e.

$$\max_{1\leq i\leq m+1} \{a^{i\cdot}x^2 - b_i\} = \min_{x\in R^n} \max_{1\leq i\leq m+1} \{a^{i\cdot}x - b_i\} = \rho_1 \leq 0$$



Assume that $x^1, x^2, \ldots, x^k$ were already calculated. Then $x^{k+1}$ is Chebyshev center of the system

$$a^{i\cdot}x \leq b_i, \ i = 1, 2, \ldots, m, m+1, \ldots, m + k$$

where

$$a^{m+k,\cdot} = \partial f(x^k), \ b_{m+k} = \partial f(x^k)x^k,$$

i.e.

(6) $$\max_{1\leq i\leq m+k} \{a^{i\cdot} x^{k+1} - b_i\} = \min_{x\in R^n} \max_{1\leq i\leq m+k} \{a^{i\cdot}x - b_i\} = \rho_k \leq 0.$$

By introducing a new variable $x_{n+1}$, the min-max problem (6) can be reduced to the following equivalent linear program :

$$\rho_k = \min \{x_{n+1} | a^{i\cdot}x - b_i \leq x_{n+1}, \ i = 1, 2, \ldots, m + k\}.$$

Number $-\rho_k$ is called Chebyshev radius of the system [8]

(7) $$a^{i\cdot}x \leq b_i, \ i = 1, 2, \ldots, m + k.$$

In [8] the following convergence theorem for the Chebyshev centers method is proved.

THEOREM 1. *If the Chebyshev radius of the system* (7) *tends to zero, i.e.*

$$\min_{x\in R^n} \max_{1\leq i\leq m+k} \{a^{i\cdot} x - b_i\} = \rho_k \to 0,$$

*then there is a subsequence* $(x^{k_i})$ *of* $(x^k)$ *such that*

$$f(x^{k_{i-1}}) \geq f(x^{k_i}), \ i = 1, 2, \ldots,$$

*and each limit point of* $(x^{k_i})$ *is an optimal solution to the problem* (5).

THEOREM 2. *The algorithm of Chebyshev centers applied to the problem* (3) *is always convergent to the optimal solution of* (3).

In the proof of this theorem we will use :

Lemma. *For each* $x^0 \in \Omega_0 = \{x \in R^n | Ax \leq b\}$, *the set*

(8) $$M = \{x \in \Omega_0 | F(x) \leq F(x^0)\},$$

*where*

$$F(x) = \sum_{i=1}^{m} (b_i - a^{i\cdot}x)^p, \ p > 1$$

*is bounded.*

*Proof.* Assume that $M$ is not bounded. Then since $M$ is convex, it has at least one recessive direction, i.e. there are $y \in \Omega_0$ and $d \in R^n$, $d \neq 0$, such that $y + \lambda d \in M$, $\forall \lambda \geq 0$.

Because $y + \lambda d \in \Omega_0$, it follows:

$$b_i - a^{i\cdot}y - \lambda a^{i\cdot}d \geq 0,\ i = 1, 2, \ldots, m,\ \forall \lambda \geq 0,$$

i.e. $a^{i\cdot}d \leq 0$, $i = 1, 2, \ldots, m$, and clearly there is $i_0 \in \{1, 2, \ldots, m\}$ such that $a^{i_0\cdot}d < 0$ (since rank $A = n$).

Now if we consider $x = y + \lambda d$, then we have

$$F(x) = \sum_{i=1}^{m}(b_i - a^{i\cdot}x)^p = \sum_{i=1}^{m}(b_i - a^{i\cdot}y - \lambda a^{i\cdot}d)^p =$$

$$= \lambda^p\left(\sum_{i=1}^{m}\left(\frac{b_i - a^{i\cdot}y}{\lambda} - a^{i\cdot}d\right)^p\right) > F(x^0)$$

for $\lambda > 0$ sufficiently large, which contradicts the fact that $x \in M$.

*Proof of Theorem 2.* From Lemma it follows that $M$ is a convex compact set for each $p > 1$. To prove the convergency of the method of Chebyshev centers it is sufficient to show, according Theorem 1, that the sequence of Chebyshev centers tends to zero.

Assume the contrary, then there is $\rho < 0$ such that $\rho_k \leq \rho$, $\forall k, \in N$.

But from the definition of the subgradient $\partial F(x)$ and $\rho_k$, we have

$$\partial F(x^i)(x^j - x^i) = a^{m+i,\cdot}(x^j - x^i) \leq \rho_i \leq \rho < 0,\quad \forall j > i,$$

or

$$-\partial F(x^i)(x^j - x^i) \geq -\rho > 0,\ \forall j > i.$$

So, since $\partial F(x)$ is obviously bounded on the compact set $M$, we have

$$0 < -\rho \leq |-\partial F(x^i)(x^j - x^i)| \leq \| \partial F(x^i)\| \cdot \|x^j - x^i\| \leq$$
$$\leq K\|x^j - x^i\|,\ \forall j > i.$$

Therefore for each subsequence $(i_k)$

$$\|x^{i_p} - x^{i_q}\| \geq \frac{-\rho}{K} > 0,\ p > q$$

which is a contradiction, because $x^k \in M$, $k \in N$, and $M$ is a compact set.

This means that $\rho_k \to 0$, $k \to \infty$, and from Theorem 1, it follows that the method of Chebyshev centers is convergent to the optimal solution to (3).



*Remark* 1. Since in our case $(1 < p < \infty)$ the function $F$ is differentiable at each point, we have

$$\partial F(x) = \nabla F(x) = \text{grad } F(x)$$

**Remark 2. As it was shown in [8] it is possible to ignore the redundante restrictions during the procedure of the algorithm in the following way. Consider a decreasing to zero sequence $(\delta_k)$ of positive numbers. If for some $i_1$ we have $\rho_{i_1} > -\delta_1$, then we define the subset:**

$$I_{i_1} \subseteq \{1, 2, \ldots, m, m+1, \ldots, i_1\}$$

for which

$$\min_{x\in\Omega_0}\ \max_{i\in I_{i_1}}\{a^{i\cdot}x - b_i\} = \rho_{i_1}$$

(9)

$$\min_{x\in\Omega_0}\ \max_{i\in I_{i_1}\setminus\{j\}}\{a^{i\cdot}x - b_i\} < \rho_{i_1},\ \ \forall j \in I_{i_1}.$$

At the next step of the algorithm we keep only restrictions $a^{i\cdot}x - b_i \leqslant 0$, for $i \in I_{i_1}$ to which we add a new restriction

$$\partial F(x^{i_1})x \leqslant \partial F(x^{i_1})x^{i_1}.$$

A new scaling of redundant restrictions will be done when we obtain $i_2 \in I_{i_1} \cup \{m + i_1, \ldots, m + i_2\}$ such that $\rho_{i_2} > -\delta_2$, and so on.

As it was shown in [8] $\delta_k = 1/\sqrt{k}$, $k \in N$ is one of the good sequences for our purpose.

*Remark* 3. Because at each iteration we add only one new restriction, in order to keep the negativity of the objective function row in the simplex tableau, it is indicated to use the dual simplex algorithm.

## 3. Statement of the algorithm

The results of the previous section are now incorporated into the following algorithm for the best one-sided $L_p$-approximation, based on the Chebyshev centers method.

*Step 0.* Compute a basic feasible solution $x^1$ of the system

(10) $$Ax \leqslant b$$

by the primal simplex algorithm. If (10) is inconsistent then terminate. Otherwise put $k = 1$, $I_k = \{1, 2, \ldots, m\}$, $\mu_k = \delta_k$, and go to Step 1.

*Step 1.* Calculate

$$a^{m+k,\cdot} = \partial F(x^k),\ b_{m+k} = \partial F(x^k)x^k.$$

*Step 2.* Find Chebyshev center $x^{k+1}$ of the system

$$a^{i\cdot}x \leqslant b_i,\ i \in I_k \cup \{m+k\}$$

by solving linear programming problem

(11) $$\rho_k = \min \{x_{n+1} | a^{i\cdot} x - x_{n+1} \leq b_i,\ i \in I_k \cup \{m+k\}\}$$

by dual simplex method.

*Step 3.* If $\rho_k \leq -\mu_k$, then $I_{k+1} = I_k \cup \{m+k\}$, $\mu_{k+1} = \delta_k$, otherwise put

$$I_{k+1} = \{i | a^{i\cdot} x^{k+1} - b_i = \rho_k\},\ \mu_{k+1} = \delta_{k+1}.$$

*Step 4.* Put $k := k + 1$ and go to Step 1.

The procedure will be continued till $|\rho_k| < \varepsilon$, for a given $\varepsilon > 0$.

## 4. Example

To illustrate the algorithm we consider the following small example: minimize

$$F(x) = (1 - x_1 - x_2)^2 + (2 + x_1 - x_2)^2 + x_1^2 + x_2^2 = \\ = 3x_1^2 + 3x_2^2 + 2x_1 - 6x_2 + 5$$

subjet to

(12) $$\begin{aligned} x_1 + x_2 &\leq 1 \\ -x_1 + x_2 &\leq 2 \\ x_1 &\leq 0 \\ -x_2 &\leq 0. \end{aligned}$$

*Step* 0. $x^1 = (0, 0)$ is a b.f.s. to (12).

$$I_1 = \{1, 2, 3, 4\},\ \mu_1 = \delta_1 = 1.$$

*Step 1.*

$$a^{5\cdot} = \nabla F(x^1) = (2, -6).$$

*Step 2.* To find Chebyshev center of the system

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ -x_1 + x_2 &\leq 2 \\ -x_1 \quad &\leq 0 \\ - \quad x_2 &\leq 0 \\ x_1 - 3x_2 &\leq 0 \end{aligned}$$

we solve the program

$$x_3 \to \min$$



subject to

$$\begin{aligned} x_1 + x_2 - x_3 &\leq 1 \\ -x_1 + x_2 - x_3 &\leq 2 \\ -x_1 \quad - x_3 &\leq 0 \\ - \quad x_2 - x_3 &\leq 0 \\ x_1 - 3x_2 - x_3 &\leq 0. \end{aligned}$$

Starting from the simplex tableau:

| | $-x_1$ | $-x_2$ | $-x_3$ | 1 |
|---|---|---|---|---|
| $y_1 =$ | 1 | 1 | $-1$ | 1 |
| $y_2 =$ | $-1$ | 1 | $-1$ | 2 |
| $y_3 =$ | $-1$ | 0 | $-1$ | 0 |
| $y_4 =$ | 0 | $-1$ | $-1$ | 0 |
| $y_5 =$ | 1 | $-3$ | $-1$ | 0 |
| $\rho =$ | 0 | 0 | $-1$ | 0 |

after five Gauss-Jordan (G—J) steps we get the tableau:

(13)

| | $-y_1$ | $-y_4$ | $-y_3$ | 1 |
|---|---|---|---|---|
| $y_2 =$ | $-1/3$ | $2/3$ | $-4/3$ | $5/3$ |
| $x_2 =$ | $1/3$ | $-2/3$ | $1/3$ | $1/3$ |
| $x_3 =$ | $-1/3$ | $-1/3$ | $-1/3$ | $-1/3$ |
| $x_1 =$ | $1/3$ | $1/3$ | $-2/3$ | $1/3$ |
| $y_5 =$ | $1/3$ | $-8/3$ | $4/3$ | $1/3$ |
| $\rho =$ | $-1/3$ | $-1/3$ | $-1/3$ | $-1/3$ |

i.e. $x^2 = (1/3, 1/3)$, $\rho_1 = -1/3$.

*Step 3.* $\rho_1 > \mu_1 = -1$, so $\mu_2 = 1/\sqrt{2}$,

$$I_2 = \{1, 3, 4\}.$$

*Step 1.*

$$a^{6\cdot} = \nabla F(x^2) = (4, -4).$$

*Step* 2. We have to find Chebyshev center $x^3$ of the system:

$$\begin{aligned} x_1 + x_2 &\leqslant 1 \\ -x_1 \quad &\leqslant 0 \\ -x_2 &\leqslant 0 \\ x_1 - x_2 &\leqslant 0. \end{aligned}$$

But from (13) it follows

$$x_1 - x_2 - x_3 = -\frac{1}{3}y_1 - \frac{4}{3}y_4 + \frac{2}{3}y_3 + \frac{1}{3} \leqslant 0.$$

So we will solve by dual simplex algorithm the problem:

| | $-y_1$ | $-y_4$ | $-y_3$ | 1 |
|---|---|---|---|---|
| $x_2 =$ | 1/3 | −2/3 | 1/3 | 1/3 |
| $x_3 =$ | −1/3 | −1/3 | −1/3 | −1/3 |
| $x_1 =$ | 1/3 | 1/3 | −2/3 | 1/3 |
| $y_6 =$ | −1/3 | −4/3 | 2/3 | −1/3 |
| $\rho =$ | −1/3 | −1/3 | −1/3 | −1/3 |

After one G–J step we get:

| | $-y_1$ | $-y_6$ | $-y_3$ | 1 |
|---|---|---|---|---|
| $x_2 =$ | 1/2 | −1/2 | 0 | 1/2 |
| $x_3 =$ | −1/4 | −1/4 | −1/2 | −1/4 |
| $x_1 =$ | 1/3 | 1/4 | −1/2 | 1/4 |
| $y_4 =$ | 1/4 | −3/4 | −1/2 | 1/4 |
| $\rho =$ | −1/4 | −1/4 | −1/2 | −1/4 |

So $x^3 = (1/4,\ 1/2)$, $\rho_2 = -1/4$.

*Step*. 3. $\rho_2 > -\mu_2 = -\delta_2 = -1/\sqrt{2}$, so $\mu_3 = 1/\sqrt{3}$, and $I_3 = \{1, 3, 6\}$.

Another approximate solution will be otained as Chebyshev center of the system

$$\begin{aligned} x_1 + x_2 &\leqslant 1 \\ -x_1 \quad &\leqslant 0 \\ x_1 - x_2 &\leqslant 0 \end{aligned}$$



to which we add

$$\partial F(x^3)(x - x^3) = 28x_1 - 24x_2 \leqslant -5.$$

Chebyshev center of this system is $x^4 = (19/77,\ 39/77) \simeq (0{,}24;\ 0{,}5)$.

The exact solution of the problem is $x^* = (0{,}1)$.

*Efficiency of the method*. The method of Chebyshev centers described in this paper was tested on some relatively small numerical examples. As it was expected, in the cases of small examples Rosen's gradient projection method is more efficient than this method. But in the cases of enough large scale problems the metod of Chebyshev centers seems to be more efficient than Rosen's algorithm.

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