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NOMOGRAMS WITH MINIMAL GLOBAL ERROR

by

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1. Let the equation

$$z_3 = f(z_1, z_2) \tag{1}$$

be represented by a nomogram N with three curvilinear scales S_1, S_2, S_3, having the following equations in orthogonal coordinates:

$$\begin{cases} x = x_i(s_i) \\ y = y_i(s_i) \end{cases} ; \; s_i = s_i(z_i) ; \; s_i'(z_i) \geqslant 0 ; \; z_i' \leqslant z_i \leqslant z_i'' ; \; i = 1, 2, 3,$$

where s_i is the arc length of the respective scale curve.

The *error of the nomogram N in the point* $P(z_i^0)$ of the scale S_i is defined by

$$E(P) = E(z_i^0) = \frac{h}{\frac{ds_i}{dz_i}(z_i^0)},$$

wher h is a constant (called the geometric error). Under the *point-wise error of the entire nomogram N* we understand the value $E_p = \max\limits_i \max\limits_{z_i' \leqslant z_i \leqslant z_i''} E(z_i)$.
A point P such that $E(P) = E_p$ is called a *point of maximal error.*

The value of z_3, found by use of the nomogram to solve equation (1), has an error, which depends on the error in the point z_3 of the scale $M_3(z_3)$, on errors in the points $M_1(z_1)$, $M_2(z_2)$ of the respective scales and on the equation (1). Thus an suitable measure for this error is the following value, associated to a the resolving line $\Delta = M_1M_2$

$$E(\Delta) = E(z_1, z_2) = \frac{\partial z_3}{\partial z_1} \cdot \frac{h}{\frac{ds_1}{dz_1}} + \frac{\partial z_3}{\partial z_2} \cdot \frac{h}{\frac{ds_2}{dz_2}} + \frac{h}{\frac{ds_3}{dz_3}} \tag{2}$$

We define the *global error of the nomogram* by

$$E_g = \max_{z_i' \leq z_i \leq z_i''} E(z_1, z_2),\ i = 1, 2.$$

A line Δ such that $E(\Delta) = E_g$ is called a *line of maximal error.*

Applying to the plane of the nomogram an admissible transformation (i.e. a collineation preserving the nomogram in the interior of a given domain, e.g. circular disc or rectangle), generally the error of the nomogram can be diminished; if not, the nomogram is said to be *optimal.*

In the papers [1], [2], [3], [4], [5], [6], [7], [8], [9] there where given criteria for a nomogram to be optimal with respect to the pointwise error and certain families of admissible transformations. In all these criteria a crucial rôle is played by the number and position of the points of maximal error.

In the present paper we shall give conditions for a nomogram N_0, with two scales on a circle C and one scale on a line D, to be *optimal with respect to the global error.* Now the number and position of the lines of maximal error will be decisive.

2. Let the origin O of the orthogonal coordinates be the centre of the unit circle C and let the Ox axis be parallel to the line D; then the equations of the scales of N_0 are

$$S_i \begin{cases} x = \cos s_i \\ y = \sin s_i \end{cases};\quad i = 1, 2 \qquad S_3 \begin{cases} x = s_3 \\ y = p \end{cases}$$

$$s_i = s_i(z_i), \qquad z_i' \leq z_i \leq z_i'',\ i = 1, 2, 3.$$

where p is the distance of the point O to the line D.

The points $M_i(s_i)$, $i = 1, 2, 3$ are collinear if and only if

$$s_3 = \frac{p(\cos s_1 - \cos s_2) + \sin (s_1 - s_2)}{\sin s_1 - \sin s_2}. \tag{3}$$

We take as admissible transformations the collinations of the plane which preserve the circle C. It is shown in [2] that such a collineation is, up to a rotation around O, a harmonic collineation with centre $O^*(\alpha, \beta)$ in the interior of the circle C and having as axis the polar of O^* with respect to C. We designate by Σ the family of these harmonic collineations, depending on two parameters α and β.

It is easily deduced that the image of the point $M(x, y)$ under $\sigma(\alpha, \beta) \in \Sigma$ is

$$x' = \frac{(\alpha^2 - \beta^2 + 1)x + 2\alpha\beta y - 2\alpha}{2\alpha x + 2\beta y - \alpha^2 - \beta^2 - 1};\quad y' = \frac{2\alpha\beta x + (\beta^2 - \alpha^2 + 1)y - 2\beta}{2\alpha x + 2\beta y - \alpha^2 - \beta^2 - 1}.$$

The image nomogram $N_{\alpha,\beta} = \sigma(N_0)$ has as scales

$$S_i' \begin{cases} x' = \dfrac{(\alpha^2 - \beta^2 + 1)\cos s_i + 2\alpha\beta \sin s_i - 2\alpha}{2\alpha \cos s_i + 2\beta \sin s_i - \alpha^2 - \beta^2 - 1} \\ y' = \dfrac{2\alpha\beta \cos s_i + (\beta^2 - \alpha^2 + 1)\sin s_i - 2\beta}{2\alpha \cos s_i + 2\beta \sin s_i - \alpha^2 - \beta^2 - 1} \end{cases},\quad i = 1, 2$$

$$S_3' \begin{cases} x' = \dfrac{(\alpha^2 - \beta^2 + 1)s_3 + 2\alpha\beta p - 2\alpha}{2\alpha \cos s_3 + 2\beta \sin s_3 - \alpha^2 - \beta^2 - 1} \\ y' = \dfrac{2\alpha\beta s_3 + (\beta^2 - \alpha^2 + 1)p - 2\beta}{2\alpha s_3 + 2\beta p - \alpha^2 - \beta^2 - 1} \end{cases}.$$

Since $\sigma(0,0)$ is a simmetry with regard to O, the nomogram $N_{0,0}$ may be identified with the given one, N_0. Direct calculations yield

$$\begin{cases} \dfrac{ds_i'}{ds_i} = \sqrt{\left(\dfrac{dx'}{ds_i}\right)^2 + \left(\dfrac{dy'}{ds}\right)^2} = \dfrac{\alpha^2 + \beta^2 - 1}{2\alpha \cos s_i + 2\beta \sin s_i - 1 - \alpha^2 - \beta^2};\quad i = 1, 2 \\ \dfrac{ds_3'}{ds_3} = \sqrt{\left(\dfrac{dx'}{ds_3}\right)^2 + \left(\dfrac{dy'}{ds_3}\right)^2} = \dfrac{\alpha^2 + \beta^2 - 1}{(2\alpha \cos s_i + 2\beta \sin s_i - 1 - \alpha^2 - \beta^2)^2}\cdot \\ \cdot \sqrt{(\beta^2 - \alpha^2 + 1 - 2\beta p)^2 + 4\alpha^2(p - \beta)^2}. \end{cases} \tag{4}$$

The expression of the error corresponding to the nomogram $N_{\alpha,\beta}$ and the resolving line Δ going through $M_1(z_1)$, $M_2(z_2)$ becomes by (3)

$$E(\Delta, \alpha, \beta) = E(z_1, z_2, \alpha, \beta) = \frac{h}{\dfrac{ds_3}{dz_3}} \left(\frac{\partial s_3}{\partial s_1} \cdot \frac{1}{\dfrac{ds_1'}{ds_1}} + \frac{\partial s_3}{\partial s_2} \frac{1}{\dfrac{ds_2'}{ds_2}} + \frac{1}{\dfrac{ds_3'}{ds_3}} \right). \tag{5}$$

Let $\Delta_1, \ldots, \Delta_n$ be the lines of maximal error of the nomogram N_{α_0,β_0}. According to a result proved in paper [9], we have:

THEOREM 1. *a) A necessary condition for* $E_g(\alpha, \beta) = \max\limits_{z_1, z_2} E(z_1, z_2, \alpha, \beta)$ *to admit a relative minimum at* (α_0, β_0) *is the existence of a positive solution* ($\forall\, j,\ t_j \geq 0$, and $\exists\, k,\ t_k > 0$) *to the system of equations*

$$\begin{cases} \sum\limits_{j=1}^{n} \dfrac{\partial E}{\partial \alpha}(\Delta_j, \alpha_0, \beta_0)t_j = 0 \\ \sum\limits_{j=1}^{n} \dfrac{\partial E}{\partial \beta}(\Delta_j, \alpha_0, \beta_0)t_j = 0. \end{cases} \tag{6}$$

b) If $n = 3$, the rank of (6) is 2 and it has a strictly positive solution ($t_1, t_2, t_3 > 0$), then $E_g(\alpha, \beta)$ admits a weak minimum at (α_0, β_0), i.e. the restriction of $E_g(\alpha, \beta)$ to an arbitrary line through (α_0, β_0) admits a relative minimum at (α_0, β_0).

c) *If $n > 3$ and if we may select three lines Δ_i with properties b), then again $E_g(\alpha, \beta)$ has a weak minimum at (α_0, β_0).*

Looking for conditions that the initial nomogram N_0 be (weakly) optimal we take $\alpha_0 = \beta_0 = 0$.

One deduces from (4)

$$(7)\qquad \begin{cases} \dfrac{\partial}{\partial\alpha}\left(\dfrac{ds_i}{ds_i'}\right)\Big|_{\alpha=\beta=0} = -2\cos s_i \\ \dfrac{\partial}{\partial\beta}\left(\dfrac{ds_i}{ds_i'}\right)\Big|_{\alpha=\beta=0} = -2\sin s_i \end{cases} \quad i = 1, 2; \quad \begin{cases} \dfrac{\partial}{\partial\alpha}\left(\dfrac{ds_3}{ds_3'}\right)\Big|_{\alpha=\beta=0} = 4s_3 \\ \dfrac{\partial}{\partial\beta}\left(\dfrac{ds_3}{ds_3'}\right)\Big|_{\alpha=\beta=0} = 2p \end{cases}$$

and from (3) we have

$$(8)\qquad \frac{\partial s_3}{\partial s_1} = \frac{\sin s_2 - p}{2\cos^2\dfrac{s_1 + s_2}{2}}; \qquad \frac{\partial s_3}{\partial s_2} = \frac{\sin s_1 - p}{2\cos^2\dfrac{s_1 + s_2}{2}}.$$

By use of (7) and (8), we get from (5)

$$(9)\qquad \begin{cases} \dfrac{\partial E}{\partial\alpha}\Big|_{\alpha=\beta=0} = -\dfrac{2h}{\dfrac{ds_3}{dz_3}} \dfrac{\sin\dfrac{s_1 + s_2}{2} - p\left(\cos\dfrac{s_1 - s_2}{2} - 2\sin\dfrac{s_1 + s_2}{2}\right)}{\cos\dfrac{s_1 + s_2}{2}} \\ \dfrac{\partial E}{\partial\beta}\Big|_{\alpha=\beta=0} = -\dfrac{2h}{\dfrac{ds_3}{dz_3}} \dfrac{\sin s_1 \sin s_2 - p\left(\sin\dfrac{s_1 + s_2}{2}\cos\dfrac{s_1 - s_2}{2} + \cos^2\dfrac{s_1 + s_2}{2}\right)}{\cos^2\dfrac{s_1 + s_2}{2}}. \end{cases}$$

To simplity the expression (8) we introduce the non-homogeneous line coordinates (u, v) of the resolving line Δ going through the points $M_i(\cos s_i, \sin s_i)$, $i = 1, 2$. Since the equation of the resolving line Δ is

$$(\sin s_1 - \sin s_2)x + (\cos s_2 - \cos s_1)y + \sin(s_2 - s_1) = 0,$$

we have

$$(10)\qquad u = -\frac{\cos\dfrac{s_1 + s_2}{2}}{\cos\dfrac{s_1 - s_2}{2}}; \qquad v = -\frac{\sin\dfrac{s_1 + s_2}{2}}{\cos\dfrac{s_1 - s_2}{2}}.$$

After some calculations, we get

$$\frac{\partial E}{\partial\alpha}\Big|_{\alpha=\beta=0} = \frac{2h}{\dfrac{ds_3}{dz_3}\,u^2} u[(1 + 2p)v + 2 + p];$$

$$\frac{\partial E}{\partial\beta}\Big|_{\alpha=\beta=0} = -\frac{2h}{\dfrac{ds_3}{dz_3}\,u^2}[-u^2(1 + p) + pv + 1].$$

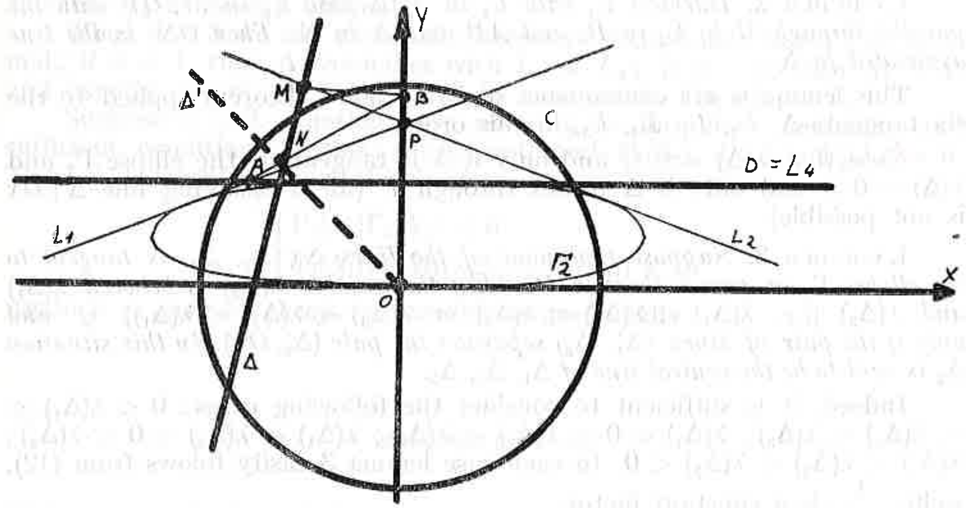
The equations

$$\Gamma_1(u, v) = \Gamma_1(\Delta) = u[(1 + 2p)v + 2 + p] = 0;$$

$$\Gamma_2(u, v) = \Gamma_2(\Delta) = -u^2(p + 1) + pv + 1 = 0$$

define in line coordinates two curves of second class, Γ_1 and Γ_2. Γ_1 is reducible and represents the pair of points $P\left(0, \dfrac{1 + 2p}{2 + p}\right)$ and the improper point of the Ox axis, i.e. Γ_1 consist of the lines through P and those parallel to Ox. The curve Γ_2 is the ellipse

$$\frac{x^2}{p + 1} + \frac{(y - p/2)^2}{(p/2)^2} = 1.$$



The following lemma is easily proved:

L e m m a. 1 *The line $ux + vy + 1 = 0$ intersects the ellipse Γ_2 in two real distinct points if and only if $\Gamma_2(u, v) < 0$.*

In the subsequent discussion we need the following pencil of line conics

$$(11)\qquad \Gamma_1(u, v) - \lambda\Gamma_2(u, v) = 0.$$

They are tangent to four fixed lines L_1 and L_2, the tangents to the ellipse Γ_2 drawn from the point P, L_3, the Ox axis, and L_4, the other tangent to Γ_2 parallel with Ox. In order to obtain an interpretation for the parameter λ we determine the second tangent T_λ to the curve (11) passing through the origin O. Using homogeneous line coordinates equation (11) writes:

$$u[(1+2p)v+(2+p)w]+\lambda[(1+p)u^2-pvw-w^2]=0;$$

a line (u, v, w) passes through O if and only if $w=0$; thus the tangents through O are the solutions of the system

$$u[(1+2p)v+\lambda(1+p)u]=0,\ w=0$$

and the line coordinates of T_λ are $u=1+2p$, $v=-\lambda(1+p)$, $w=0$. Denoting be m_λ the slope of T_λ, we have

$$\lambda=\frac{1+2p}{1+p}\,\frac{1}{m_\lambda}. \tag{12}$$

If Δ is any line different from L_i, $i=1,\ldots,4$, there exist a unique curve (11) tangent to Δ; we denote the corresponding value of the parameter λ by $\lambda(\Delta)$ and the tangent $T_{\lambda(\Delta)}$ by Δ' and call it the *line associated to* Δ. We can construct Δ as follows:

Lemma 2. *Intersect L_1 with L_4 in A, Δ with L_2 in M, OP with the parallel through M to L_3 in B, and AB with Δ in N. Then ON is the line associated to Δ.*

This lemma is a a consequence of Brianchon's theorem applied to the six tangents Δ, L_2, L_1, L_4, L_3, in this order.

Note that $\lambda(\Delta)=\infty$ if and only if Δ is tangent to the ellipse Γ_2 and $\lambda(\Delta)=0$ if and only if Δ passes through P (for a resolving line $\Delta \parallel Ox$ is not possible).

Lemma 3. *Suppose that none of the lines Δ_1, Δ_2, Δ_3, is tangent to the ellipse Γ_2 or passes through P. Then the number $\lambda(\Delta_2)$ is between $\lambda(\Delta_1)$ and $\lambda(\Delta_3)$ (i.e. $\lambda(\Delta_1)<\lambda(\Delta_2)<\lambda(\Delta_3)$ or $\lambda(\Delta_3)<\lambda(\Delta_2)<\lambda(\Delta_1)$) if and only if the pair of lines (Δ_1', Δ_3') separates the pair (Δ_2', Ox). In this situation Δ_2 is said to be the central line of Δ_1, Δ_2, Δ_3.*

Indeed, it is sufficient to consider the following cases: $0<\lambda(\Delta_1)<$ $<\lambda(\Delta_2)<\lambda(\Delta_3)$; $\lambda(\Delta_1)<0<\lambda(\Delta_2)<\lambda(\Delta_3)$; $\lambda(\Delta_1)<\lambda(\Delta_2)<0<\lambda(\Delta_3)$; $\lambda(\Delta_1)<\lambda(\Delta_2)<\lambda(\Delta_3)<0$. In each case lemma 3 easily folows from (12), wehre $\frac{1}{m_\lambda}$, is a constant factor.

3. To apply theorem 1. we consider the following *main case*: there are exactly 3 lines of maximal error, say Δ_1, Δ_2, Δ_3, $\Gamma_i(\Delta_j)\neq 0$, $i=1,2$; $j=1,2,3$, and the associated lines Δ_1', Δ_2', Δ_3' are distinct.

Then the system (6) becomes

$$\Gamma_i(\Delta_1)t_1+\Gamma_i(\Delta_2)t_2+\Gamma_i(\Delta_3)t_3=0;\quad i=1,2. \tag{13}$$

If Δ_2 is the central line among Δ_1, Δ_2, Δ_3, and

$$\lambda_i=\lambda(\Delta_i)=\frac{\Gamma_1(\Delta_i)}{\Gamma_2(\Delta_i)},$$

the system of equation (13) has a positive solution if and only if the numbers

$$\frac{\lambda_1-\lambda_2}{\Gamma_2(\Delta_3)},\quad \frac{\lambda_2-\lambda_3}{\Gamma_2(\Delta_1)},\quad \frac{\lambda_3-\lambda_1}{\Gamma_2(\Delta_2)} \tag{14}$$

have same sign, that is to say when $\Gamma_2(\Delta_1)$, $-\Gamma_2(\Delta_2)$, $\Gamma_2(\Delta_3)$ have equal sign. Using also lemma 1. we can state:

THEOREM 2. *In the main case the following condition is necessary and sufficient for the nomogram N_0 to be weakly optimal:*

(15) *Either exactly one of the three lines of maximal error is a secant to the ellipese Γ_2 and this the central one, or exactly two lines of maximal error are secant to Γ_2 and these are not central.*

Note that lemmas 2 and 3 permit a graphical verification of condition (15).

In case when the number n of the lines of maximal error is greater than 3 and if we can select 3 of them satisfying the conditions of theorem 2, then N_0 is weakly optimal.

When $n<3$ we can state only necessary conditions: Let N_0 be optimal; if $n=1$, then Δ_1 coincides with L_1 or L_2; if $n=2$, then $\Delta_1'=\Delta_2'$ and exactly one of the lines Δ_1, Δ_2 is a secant to the ellipse Γ_2.

Suppose $n=3$. Another case allowing to formulate necessary and sufficient conditions is this: $\Gamma_2(\Delta_3)=0$ and $\Gamma_1(\Delta_3)$, $\Gamma_2(\Delta_1)$, $\Gamma_2(\Delta_2)\neq 0$. Then N_0 is weakly optimal if only if

$$\begin{cases}\Gamma_2(\Delta_1)\Gamma_2(\Delta_2)<0\\ \Gamma_2(\Delta_1)\Gamma_1(\Delta_3)(\lambda(\Delta_1)-\lambda(\Delta_2))<0\end{cases}$$

Again it is easy to check the conditions graphically.

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