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ON AN INTEGRO-DIFFERENTIAL SINGULAR EQUATION

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In this paper we consider the problem of the existence and the uniqueness of the solution in a set $\mathcal{V}$ of the singular integro-differential equation (1) with the initial value condition (2).

Such equations appear in fluid dynamics, namely in the determination of jet lines [1]. The problem is reduced to Poincaré boundary problem (5) for a harmonic function W.

The final results are included in the theorem (*). The two cosequences reffer to the weak singular Fredholm equation (6).

1. Let us consider the following integro-differential equation with a singularity of Cauchy type

$$f(t) \cdot V'(t) + \frac{1}{\pi} \oint_{-\infty}^{+\infty} \frac{V(x)}{t-x} \mathrm{d}x = g(t) \tag{1}$$

to which we join an initial condition of the type

$$V(0) = V^0, \; V^0 \in \mathbf{R}. \tag{2}$$

The given functions $f(t)$ and $g(t)$ are supposed analytical on the real axis, the first of them satisfying also $f(t) > 0$ on $\mathbf{R}$ while the solution $V(x)$ belongs to the set $\mathcal{V}$ of the real, analytical functions, Hölderian in all the finite points of the axis and satisfying the „P–R conditions"[1]

$$\lim_{|x|\to\infty} V(x) = 0, \quad |V(x)| < \frac{A}{|x|^{\mu}}$$

where A and μ are positive constants.

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[1] We call these conditions „*P–R conditions*" after the name of PRZEWORSKA–ROLEWICZ, who has given them when studying the existence, in the sense of the principal value, of the integral $\int_{-\infty}^{+\infty} \frac{V(x)}{t-x} \mathrm{d}x$ [2].

For the study of the existence and the uniqueness of the problem (1) + (2) we shall first take into account the correspondent homogenous equation:

(3) $$f(t)\cdot V'(t)+\frac{1}{\pi}\oint_{-\infty}^{+\infty}\frac{V(x)}{t-x}\,dx=0$$

to which we attach the following boundary problem:

„Determine a function $G(\zeta)=U-iW$, holomorphic in the upper half-plane, whose real part U is a uniform bounded function which fulfils the boundary condition

$$U(x, y)|_{0x}=V(x),$$

where $V(x)$ is the solution, supposed to exist, of the homogenous integro-differential equation (3)".

The solution of the above boundary problem, up to an additive pure imaginary constant, is

$$G(\vartheta)=-\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{V(x)}{x-\xi}\,dx$$

the existence conditions, in the sense of the principal value, of this integral of Cauchy type beeing fulfiled by the above demands requiried from $V(x)$ [2].

Therefore the function W will tend to a well determined limit when the point ζ tends to the point t of the real axis, provided that the function $V(x)$ satisfies generally enough conditions, as for instance a Hölder condition which is assured by hypothesis itself. More precisely we shall have as limit value

$$W(t)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(x)}{x-t}\,dx,$$

where the existence, in the sense of the principal value, of this integral is also assured by the same conditions imposed on $V(x)$. Now, taking into account that on the whole real axis $\frac{\partial U}{\partial x}=V'(x)$, the homogenous integro-differential aquation (3) express in fact the fulfilment of the following condition on the whole boundary of the upper half-plane, the domain of holomorphicity of the function $G=U-iW$,

(3′) $$f(t)\,\frac{\partial U}{\partial t}=-W(t),\qquad \forall t\in\mathbf{R}.$$

But the above equality shows that due to the analyticity of the function $U=V(x)$ on the real axis, we shall have the same property for the function W, too. So the restriction on the Ox, of the searched function G, beeing also analytical, it will transform this axis — the boundary of the domain of holomorphicity into an analytical curve, property which assures the validity on the boundary Ox of the Cauchy — Riemann conditions. Finally, by $\frac{\partial U}{\partial x}=-\frac{\partial W}{\partial y}=-\frac{dW}{dn}$ in the points of the real axis, we see that the homogenous singular integro-differential equation is equivalent to the fulfilment of the following boundary conditions on the real axis

(4) $$f(t)\cdot\frac{dW(t)}{dn}-W(t)=0,\qquad t\in(-\infty,+\infty)$$

So the problem of the existence of the solution of the singular integro-differential equation (3) is connected with that of the determination of a function W, harmonic in the upper half-plane, analytical on the real axis, which fulfils on this axis the previous mixed boundary condition (4). But this new problem is a homogenous boundary problem of the *Poincaré type*. Using now a Green's formula for harmonic functions[2], more precisely

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty}(\operatorname{grad}W)^2\,dxdy=-\int_{-\infty}^{+\infty}W\,\frac{dW}{dn}\,dx,$$

where $\frac{dW}{dn}$ will be taken from (4), and integrating by parts, we obtain

$$0\leq\int_{-\infty}^{+\infty}\int_{0}^{+\infty}(\operatorname{grad}W)^2dxdy=-\int_{-\infty}^{+\infty}\frac{1}{f(x)}W^2dx\leq 0.$$

Hence $\frac{\partial W}{\partial x}=\frac{\partial W}{\partial y}=0$, so the function W is necessarily a constant. As

[2] From the Green's formula for a bounded domain

$$\iint_{D_R}(\operatorname{grad}W)^2\,dxdy=-\int_{-R}^{R}W\,\frac{dW}{dn}\,dx-\int_{C_R}W\,\frac{dW}{dn}\,ds,$$

where $D_R(0;R)$ is the demi-circle in the upper half-plane, if $R\to\infty$ then, making use of the boundness of $-\int_{-\infty}^{+\infty}W\frac{dW}{dn}$ ($\leqslant 0$ from (4)] and of the behaviour of the first derivatives of the bounded function W in the neighbourhood of infinity $\left(\text{as }\frac{1}{R^2}\right)$, we obtain the validity of the above Green's formual for unbounded domains.

$f(t) > 0$, we have by the same condition (4), that the constant value of W must be zero.

Therefore we have proved that the only solution of the attached homogenous Poincaré's problem is identically zero. Since the so called „index" χ of the problem, which now can be expressed by the formula $\chi = \frac{1}{\pi}[\arg 1]_{0\chi}$ vanishes, the solution of the nonhomogenous Poincaré problem, for every right sides $\frac{g(t)}{f(t)}$, namely

$$\frac{dW(t)}{dn} - \frac{1}{f(t)} W(t) = \frac{g(t)}{g(t)}, \quad \forall t \in \mathbf{R}, \tag{5}$$

exists and it is unique [3].

But this nonhomogenous boundary problem corresponds in fact to the initial nonhomogenous singular integro- differential equation (1). It comes out that the searched solution $V(x)$ of this equation (1) will be the restriction on the real axis of the harmonic function U, harmonical conjugate of the unique solution W of the boundary problem (5). Obviously that function $V(x)$, solution of (1), will be determined up to an additive constant, so we may give the following final theorem :

* THEOREM. *For any analytical real functions $f(t)$ and $g(t)$, where $f(t)>0$, the nonhomogenous singular integro-differential equation (1) with the initial value condition (2) has, in the set V of the real analytical functions, Hölderian on real axis, satisfying the P—R conditions, a solution and only one.*

2. The problem (1) + (2) could be reduced to an integral equation of Fredholm type with weak singularity. Indeed, putting

$$\int_0^t \frac{f(q)}{q-x}\,dq = K(t, x), \int_0^t g(g)dq + V^0 = h(t)$$

and integrating the equation (1) with the condition (2) from 0 to t, we arrive at the following singular integral equation of the Fredholm type[3]

$$V(t) + \frac{1}{\pi}\int_{-\infty}^{+\infty} V(x)K(t, x)dx = h(t) \tag{6}$$

[3] The permutation of the two integrals which occur is possible since the following limit is uniform with regard to t.

$$\lim_{\substack{A\to\infty \\ \varepsilon\to 0}} \left[\int_{-A}^{t-\varepsilon} \frac{V(x)}{t-x}\,dx + \int_{t+\varepsilon}^{A} \frac{V(x)}{t-x}\,dx\right]$$

Due to the fact that for this integral equation with weak singularity of logarithmic type the Fredholm's alternative is applicable, we may formulate the following consequences of the theorem just prooved.

Consequence I: „The integral operator $A : V(x) \to \int_{-\infty}^{+\infty} V(x)k^*(x, t)dt$ *defined on the set* $\mathcal{V}$ *with the values in the same set, has a single fixed point and only one, namely* $V(x) \equiv 0$."

$$\left(\text{We denoted } k^*(x, t) = -k(x, t) = \int_0^x \frac{f(q)}{t-q}\,dq\right)$$

Consequence II: „For any real analytical functions $f(t)$ and $g(t)$, where $f(t) > 0$, the integral equations of Fredholm type with weak singularity (6) *has a solution and only one in the same set V".*

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