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ON UNIFORMLY EQUICONTINUOUS DYNAMICAL SYSTEMS

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1. M. BEBUTOV defined in [1] the shift system τ on $C(\mathbf{R}, \mathbf{R})$ which was much studied later in the mathematical literature. He used the compact-open topology on $C(\mathbf{R}, \mathbf{R})$. It is natural to look for such metrics (or topologies) on $C(\mathbf{R}, \mathbf{R})$ in respect to which τ becomes an uniformly continuous (equicontinuous, respectively, uniformly equicontinuous) dynamical system (s. [3]). The first temptation is to use the uniform metric, but in respect to it τ is not, at least, continuous.

In this paper, we obtain τ as uniformly equicontinuous dynamical system using for $C(\mathbf{R}, \mathbf{R})$ the metric S which we defined in [8]. We call it ue-Bebutov system and study an universality property of them. Also we compare the usual and the ue-Bebutov systems.

2. We recall some notations and definitions from topological dynamics. Let X be a metric space with a fixed metric d.

D e f i n i t i o n 1. *A dynamical system (respectively a uniformly equicontinuous dynamical system) on X is a function $\pi : X \times \mathbf{R} \to X$ that satisfies the following axioms:*

(1) $$\pi(x, 0) = x, \text{ for every } x \text{ in } X$$

(2) $$\pi(\pi(x, t), s) = \pi(x, t + s), \text{ for } x \text{ in } X,\ t \text{ and } s \text{ in } \mathbf{R}$$

(3) π *is continuous (respectively uniformly continuous).*

For a fixed point x in X we use the following notations:

(4) $$\gamma^+(x) = \{y \in X : \exists\ t \geqslant 0,\ y = \pi(x, t)\}$$

(5) $$L^+(x) = \{y \in X : \exists\ t_n \to +\infty,\ y = \lim \pi(x, t_n)\}$$

(6) $$J^+(x) = \{y \in X : \exists\ x_n \to x,\ \exists\ t_n \to +\infty,\ y = \lim \pi(x_n, t_n)\}$$

to denote the *positive trajectory*, the *positive limit set*, respectively the *positive prolongational limit set of x*.

D e f i n i t i o n 2. *A point* $x \in X$ *is called*:

a) *positively Lagrange stable if* $\gamma^+(x)$ *is relatively compact*

b) *positively Poisson stable if* $x \in L^+(x)$

c) *positively nonwandering if* $x \in J^+(x)$.

D e f i n i t i o n 3. *A set* $M \subseteq X$ *is called*:

a) *positively invariant if for any* $t \geq 0$, $\pi(M, t) \subseteq M$

b) *positively Lyapunov stable if it is positively invariant and for any* $\varepsilon > 0$ *there is a* $\delta > 0$, *such that for each* y *in* X *with* $d(y, M) = \inf\{d(y, x): x \in M\} < \delta$, *and for each* $t \geq 0$ *we have* $d(\pi(y, t), \pi(M, t)) < \varepsilon$.

Convention. For all of the above notations and definitions we have a „negative" and a global version taking $t \leq 0$, respectively $t \in \mathbf{R}$. If the dynamical system is uniformly equicontinuous we add a lower index „u" in the notations and the adjective „uniform" in definitions.

D e f i n i t i o n 4. *Let* π *and* σ *be dynamical systems defined on* X *respectively on* Y. *They are isomorphic (respectively uniformly isomorphic) if there exists a homeomorphism* $h: X \to Y$ *(with* h *and* h^{-1} *uniformly continuous) such that the following diagram*:

$$\begin{array}{ccc} X \times \mathbf{R} & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle h \times 1_R} & & \downarrow{\scriptstyle h} \\ Y \times \mathbf{R} & \xrightarrow{\sigma} & Y \end{array}$$

is comutative. If π *is isomorphic (uniformly isomorphic) with a subsystem of* σ, *one says that* π *is embeddable (uniformly embeddable) in* σ. If *the bijection* h *is only continuous, one says that* π *may be applied on* σ.

3. Let $C(\mathbf{R}, X)$ be the set of all continuous functions from $\mathbf{R}$ to X. To define some metrics for $C(\mathbf{R}, X)$ we shall use the function $I: [0, \infty] \to [0, 1]$ given by:

$$(7) \qquad I(t) = \begin{cases} \dfrac{t}{t+1} & \text{for } t \in [0, \infty) \\ 1 & \text{for } t = \infty \end{cases}$$

The uniform metric T for $C(\mathbf{R}, X)$ may be defined by:

$$(8) \qquad T(f, g) = I(\sup\{d(f(t), g(t)) : t \in \mathbf{R}\}).$$

The compact-open topology on $C(\mathbf{R}, X)$ is generated by the metric K defined as follows:

$$(9) \qquad K(f, g) = \sum_{n=1}^{\infty} 2^{-n} \cdot I(\max\{d(f(t), g(t)) : |t| \leq n\}).$$

In [8] we defined a Pompeiu — Hausdorff type metric by:

$$(10) \qquad S(f, g) = I(\sup\{S_0(f, g), S_0(g, f)\})$$

where

$$(10') \quad S_0(f, g) = \inf\{r > 0 : \forall t \in \mathbf{R}, \ \inf\{d(f(t), g(s)) : |t - s| < r\} < r\}$$

with the usual convention: $\inf \varnothing = \infty$.

Using the metric K for $C(\mathbf{R}, X)$, M. BEBUTOV studied in [1] (for $X = \mathbf{R}$) the shift $\tau: C(\mathbf{R}, X) \times \mathbf{R} \to C(\mathbf{R}, X)$ defined by:

$$(11) \qquad \tau(f, t) = f_t, \text{ where } f_t(s) = f(t + s).$$

He proved that τ is a dynamical system.

If one uses the metric T for $C(\mathbf{R}, X)$, the shift τ is generally discontinuous. For example, if X is a normed space and $x \neq 0$ is a fixed element in X, τ is discontinuous in f defined by $f(t) = \exp(t) \cdot x$.

Let us denote by τ_u the function τ defined by (11) on $C(\mathbf{R}, X)$ metrized by S.

THEOREM 1. τ_u *is an uniformly equicontinuous dynamical system.*

Proof. Let $r > 0$ and suppose $S(f, g) < r/2$ and $|t - s| < r/2$. For any $\alpha \in \mathbf{R}$ there is an β in $\mathbf{R}$ such that $|\alpha + t - \beta| < r/2$ and $d(f(\alpha + t), g(\beta)) < r/2$. For $u = \beta - s$ we have:

$$|\alpha - u| = |\alpha - \beta + s| \leq |\alpha + t - \beta| + |s - t| < r$$

and

$$d(f_t(\alpha), g_s(u)) < r/2 < r.$$

Hence

$$S_0(f_t, g_s) < r$$

and interchanging f and g we get:

$$S(f_t, g_s) < r.$$

D e f i n i t i o n 5. *We call* τ_u *ue-Bebutov system on* X.

Remark 1. It was proved in [8] that the metric S is finer than K, that is the identity function

$$i: (C(\mathbf{R}, X), S) \to (C(\mathbf{R}, X), K)$$

is continuous. It follows that the ue-Bebutov system may be applied on Bebutov system τ. As an easy consequence we obtain the following:

THEOREM 2. *For each* $f \in C(\mathbf{R}, X)$ *we have*:

a) $L_u^+(f) \subseteq L^+(f)$

b) *If* f *is uniformly positively Lagrange stable then it is positively Lagrange stable.*

c) *If* f *is uniformly positively Poisson stable it is positively Poisson stable.*

d) *If* f *is uniformly positively nonwandering, it is positively nonwandering.*

Remark 2. As concerns Lyapunov stability, the implication is converse. More exactly, every positively invariant set is uniformly positively Lyapunov stable.

All these statements remain true if we replace „positive'' by „negative''.

Generally the above inclusions and implication are strictly even in the case $X = \mathbf{R}$ as one see in following examples.

Example 1. Let us consider the function:

$$f(t) = \begin{cases} 1 - |t| & \text{for } |t| \leqslant 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

The nul application belongs to $L^+(f)$ and to $L^-(f)$ but not to $L_u^+(f) \cup L_u^-(f)$. Moreover $L_u(f) = \emptyset$, hence (s. [6]) f is not uniformly Lagrange stable (in any sense). However it is positively (and negatively) Lagrange stable.

Example 2. Let us construct the double infinite sequence $(a_n)_{n=-\infty}^{\infty}$ as follows: we start with the arbitrary non-constant sequence $(a_n)_{n=0}^{\infty}$ (for example $a_n = n$). Then, step by step, for $n = 0, 1, 2, \ldots$ we put the block $a_{-n}, \ldots, a_0, \ldots, a_n$ as the block $a_{-(n+1)^2}, \ldots, a_{n^2+n+1}$. Then we define the function $f: \mathbf{R} \to \mathbf{R}$ by:

$$f(t) = (t - n)(a_{n+1} - a_n) + a_n \text{ for } t \in [n, n+1),\ n \in \mathrm{Z}.$$

For it $K(f, f_{n^2+n+1}) \leqslant 1/n$, hence $f \in L^+(f)$. But $f \notin L_u^+(f)$ because $S(f, f_s) \geqslant 1/2$ for $s > 1$ (at least in the concrete example which was given).

Similarly, starting with a kernel of distinct numbers a_0, a_1, a_2 and constructing alternatively on the left and on the right the sequence $(a_n)_{n=-\infty}^{\infty}$, we obtain, as before, an f which is Poisson stable (in both directions) but is not uniformly Poisson stable in any direction.

Example 3. As we said, any critical point is uniformly Lyapunov stable. On the contrary, no critical point is Lyapunov stable. Indeed, if $f(t) = x_0 =$ constant and $0 < \delta < \varepsilon < 1$ are arbitrarily, the continuous function:

$$g^{\varepsilon}(t) = \begin{cases} x_0 & \text{if } t \notin (-1/\varepsilon - 1,\ -1/\varepsilon) \\ x_0 + 1 & \text{for } t = -1/\varepsilon - 1/2 \\ \text{linear} & \text{in rest} \end{cases}$$

satisfies $K(g^{\varepsilon}, f) < \delta$ but $K(g^{\varepsilon}_{1/\varepsilon+1/2}, f) = 1 > \varepsilon$.

4. For a given dynamical (or uniformly equicontinuous dynamical) system defined on a metric space X, let us denote:

$$\Pi = \{\pi_x : \mathbf{R} \to X,\ \pi_x(t) = \pi(x, t),\ \forall t \in \mathbf{R},\ x \in X\}.$$

The shift system τ (respectively τ_u) on $C(\mathbf{R}, X)$ induces on Π, which is invariant, a dynamical (respectively an uniformly equicontinuous dynamical) system. We call $\tau|\pi$ (respectively $\tau_u|\pi$) Bebutov system associated to π.

THEOREM 3. *Any dynamical (uniform equicontinuous dynamical) system is isomorphic (respectively uniformly isomorphic) with its associated Bebutov system.*

Proof. The application $i : X \to \Pi$ defined by $i(x) = \pi_x$ is a bijection and the diagram:

$$\begin{array}{ccc} X \times \mathbf{R} & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle i \times 1_R} & & \downarrow{\scriptstyle i} \\ \Pi \times \mathbf{R} & \underset{(\tau_u/\pi)}{\xrightarrow{\tau/\pi}} & \Pi \end{array}$$

is commutative. We have to prove the continuity (uniform continuity) of i and i^{-1}.

Case a) π is a dynamical system:

Let $x_n \to x$ and $\varepsilon > 0$. Fix $i_0 \in \mathbf{N}$ such that $\sum_{i=i_0+1}^{\infty} 2^{-i} < \frac{\varepsilon}{2}$. Because $\pi(x_n, t) \to \pi(x, t)$, for every $t \in \mathbf{R}$ there is a N_t such that $n > N_t$ implies $d(\pi(x_n, t), \pi(x, t)) < \varepsilon/2$. The corespondence $t \to N_t$ being continuous, we may find $N_0 = \max\{N_t : |t| \leqslant i_0\}$. Then for every $n > N_0$:

$$K(\pi_{x_n}, \pi_x) \leqslant \sum_{i=1}^{i_0} 2^{-i} I\ (\max\{d(\pi(x_n, t), \pi(x, t)) : |t| \leqslant i\}) + $$
$$+ \sum_{i=i_0+1}^{\infty} 2^{-i} \leqslant \varepsilon/2 \sum_{i=1}^{i_0} 2^{-i} + \varepsilon/2 < \varepsilon$$

that is i is continuous. i^{-1} is also continuous because

$I(d(x, y)) \leqslant K(\pi_x, \pi_y)$.

Case b) π is an uniformly equicontinuous dynamical system.

For any $\varepsilon > 0$ there is an $\delta > 0$ ($\delta < \varepsilon$) such that $d(x, y) < \delta$ and $|t - s| < \delta$ implies $d(\pi(x, t), \pi(y, s)) < \varepsilon$. That is $d(x, y) < \delta$ implies $S(\pi_x, \pi_y) < \varepsilon$, hence i is uniformly continuous. Also if $S(\pi_x, \pi_y) < \delta$ there exists $s = s(x, y)$, $|s| < \delta$ such that $d(\pi(x, 0), \pi(y, s)) < \delta$. Hence

$$d(x, y) \leqslant d(\pi(x, 0), \pi(y, s)) + d(\pi(y, s), \pi(y, 0)) < \delta + \varepsilon < 2\varepsilon$$

that is i^{-1} is uniformly continuous.

Remark 3. It is well known from [1] and [5] that the Bebutov system (on $\mathbf{R}$) has an universality property for compact systems which is easy to generalize (s. [7]) as follows: the dynamical system π defined on a compact metric space X is embeddable in Bebutov's system on the Banach space Y if and only if the set of all critical points of π is embeddable in Y, i.e. is homeomorphic with a subset of Y. Analogously, we have the following:

THEOREM 4. *The dynamical system π defined on a compact metric space X is embeddable (uniformly embeddable) in the ue-Bebutov system on Banach space Y if and only if it is uniformly equicontinuous and the set of its critical points is uniformly embeddable in Y.*

Proof. Necessity is immediately because any dynamical system which is uniformly embeddable in an uniformly equicontinuous dynamical system is uniformly equicontinuous.

Sufficiency: let us denote by Γ the set of all critical points of π and by γ an embedding of Γ in Y. By the method used in [5], it was proved in [7] that there is a continuous function $f: X \to Y$ such that $f|\Gamma = \gamma$ and for any pair of distinct points $x, y \in X$:

$$f(\pi(x, t)) \neq f(\pi(y, t)) \tag{12}$$

for at least a real t. Define the function $h: X \to (C(\mathbf{R}, Y), S)$ by $h(x) = f \cdot \pi_x$. The space X being compact, f is uniformly continuous hence for any $\varepsilon > 0$ there is $\eta < 0$ such that $\| f(x) - f(y) \| < \varepsilon$ if $d(x, y) < \eta$. By the uniform equicontinuity of π, there is δ, $0 < \delta \leq \eta$, such that $d(x, y) < \delta$ implies $d(\pi(x, t), \pi(y, t)) < \eta$ for any $t \in \mathbf{R}$. Hence $S(h(x), h(y)) < \varepsilon$, therefore h is uniformly continuous. By (12) h is injective and X being compact and $C(\mathbf{R}, Y)$ Hausdorff in respect te S, it follows that $h^{-1}: h(X) \to X$ is uniformly continuous. Consequentely π is uniformly embedded in the ue-Bebutov system on Y.

Consequence. *Any dynamical system defined on a compact metric space is uniformly embeddable in ue-Bebutov system on l^2.*

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