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RELATIONS BETWEEN CAPACITIES, HAUSDORFF h-MEASURES AND p-MODULES

by

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In some previous papers [3–7], I considered the following problem: Given a K-quasiconformal mapping $f: B \leftrightarrows D^*$ of the unit ball onto a domain (B, D^* contained in the Euclidean n-space R^n), to estimate the exceptional set E_0 of the unit sphere S with the property that the image $\gamma^* = f(\gamma)$ of any endcut γ of a point of E_0 from B is an unrectifiable arc. In [3], I proved that E_0 is compact and its conformal capacity is zero. Next, in [4–7] I decided to try to estimate E_0 by means of other kinds of capacities and by Hausdorff h-measures. In doing this, I established that E_0 is of Hausdorff h-measure $H_h(E_0) = 0$ with the measure function $h(r) = \left(\log \frac{1}{r}\right)^{-\beta}$, where $\beta > n - 1$, $r < \frac{1}{e}$, and also of Φ-capacity $C_\Phi(E_0) = 0$, where the kernel $\Phi(r) = \left(\log \frac{1}{r}\right)^{\beta}$ with $\beta > n - 1$ and $r < \frac{1}{e}$.

In the present paper, following a suggestion of L. I. HEDBERG, I shall improve these results, showing that $H_h(E_0) = 0$ with

$$(1) \qquad h(r) = \begin{cases} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{1-n} \left(\log_m \frac{1}{r}\right)^{-\beta} \text{ for } r \leq r_m, \\ \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r_m}\right)^{1-n} \left(\log_m \frac{1}{r_m}\right)^{-\beta} \text{ for } r > r_m, \end{cases}$$

$\forall$(i.e. for any) $\beta > n-1$, $\log_m \frac{1}{r_m} > 1$ and $m = 1, 2, \ldots$ and also that $C_\Phi(E_0) = 0$ with

$$(2) \qquad \Phi(r) = \begin{cases} \left(\log \frac{1}{r}\right)^{n-1} \prod_{k=2}^{m-1} \left(\log_k \frac{1}{r}\right)^{n-2} \left(\log_m \frac{1}{r}\right)^{\beta} & \text{for } r \leqslant r_m, \\ \left(\log \frac{1}{r_m}\right)^{n-1} \prod_{k=2}^{m-1} \left(\log_k \frac{1}{r_m}\right)^{n-2} \left(\log_m \frac{1}{r_m}\right)^{\beta} & \text{for } r > r_m, \end{cases}$$

$\forall \beta > n-1$ and $m = 1, 2, \ldots$.

But I obtain even more, i.e. some inclusion relations between Hausdorff h-measures and different kinds of Bessel capacities, generalizing in this way the corresponding theorems of D. ADAMS and N. MEYERS [2]. Some of the inclusions are showed to be best possible. I generalize also an important lemma of du PLESSIS [14] and finally, from the preceding relations, I deduce the corresponding inclusion relations between Hausdorff h-measures or Bessel capacities and p-modules.

Now, let us recall different concepts involved in the sequel especially Hausdorff h-measures and the different kinds of capacities.

The *p-modulus of an arc family* Γ of a domain $D \subset R^n$ is

$$M_p(\Gamma) = \inf_\rho \int \rho(x)^p \, dx$$

where dx is the volum element and the infimum is taken over all Borel measurable functions $\rho(x) \geqq 0$ such that $\int_\gamma \rho ds \geqq 1$ $\forall \gamma \in \Gamma$. The n-modulus $M(\Gamma) = M_n(\Gamma)$ is called simply *modulus*.

A homeomorphism $f: D \rightleftarrows D^*$ is said to be *K-quasiconformal* $(1 \leqq K < \infty)$ if

$$\frac{M(\Gamma)}{K} \leqslant M(\Gamma^*) \leqslant KM(\Gamma),$$

where Γ is an arbitrary arc family contained in D and $\Gamma^* = f(\Gamma)$.

The *Hausdorff h-measure* $H_h(E)$ of a set $E \subset R^n$ is the non-negative number

$$H_h(E) = \lim_{\delta \to 0} \inf_{\{E_m\}} \Sigma_m h[d(E_m)],$$

where the measure function h is supposed to be continuous, non-negative, non-decreasing in some interval $(0, r')$, $r' > 0$, and such that $\lim_{r \to 0} h(r) = 0$, and where the infimum is taken over all countable coverings $\{E_m\}$ of E by sets E_m having a diameter $d(E_m) \leqq \delta$.

The *p-capacity of a compact set* $F \subset R^n$ is given as

$$\text{cap}_p F = \inf_u \int |\nabla u(x)|^p \, dx,$$

where the infimum is taken over all $u \in C^1$, with $u_{|F} = 1$, $\nabla u = \left(\frac{\partial u}{\partial x^1}, \ldots, \frac{\partial u}{\partial x^n}\right)$ and where, in the particular case $p = n$, the support S_u of u is contained in a fixed ball (say of radius R_0). The corresponding n-capacity cap F is named the *conformal capacity*.

By a *kernel* we will mean a function $\Phi: [0, +\infty] \to [0, +\infty]$, which is non-increasing and lower semi-continuous. Corresponding to such a kernel, we set $\Phi(x) = \Phi(|x|)$ for $x \in R^n$.

The *Φ-capacity* $C_\Phi F$ of a compact set $F \subset R^n$ is characterized by

$$C_\Phi(F) = [\inf_\mu \iint \Phi(x-y) d\mu(x) d\mu(y)]^{-1}$$

where $\lim_{r \to 0} \Phi(r) = +\infty$ and the infimum is taken over all the measures $\mu \geqq 0$ with $\mu(R^n) = 1$ and the support $S_\mu \subset F$.

For $1 < p \leqq \infty$, L_p^+ will be the vector space of functions $f(x) \geqq 0$ measurable with

$$||f||_p = [\int f(x)^p \, dx]^{\frac{1}{p}} < \infty, \quad 1 < p < \infty,$$

or

$$||f||_\infty = \text{ess sup } |f(x)| < \infty, \quad p = \infty.$$

If Φ is a kernel, then, for $1 < p < \infty$ and $E \subset R^n$ we define the *capacity*

$$C_{\Phi, p}(E) = \inf_f ||f||_p^p,$$

where $f \in L_p^+$ and the convolution

$$\Phi * f(x) \geqq 1 \;\; \forall x \in E.$$

A function f which satisfies these conditions is called a *test function* for $C_{\Phi,p}(E)$. We remind that the convolution

$$\Phi * f(x) \equiv \int \Phi(x-y) f(y) dy.$$

Let $\mathfrak{M}^+(E)$ be the cone of all Radon measures $\mu \geqq 0$ carried by E, i.e. with $\mu(R^n - E) = 0$, and $L_1^+(E)$ its subspace composed of all measures μ with $||\mu||_1 \equiv$ the total variation of $\mu < \infty$.

For $\mu \in L_1^+(E)$, we have the convolution

$$\Phi * \mu(x) = \int \Phi(x-y) d\mu(y).$$

Let $\mathfrak{L}_1$ be the σ-algebra of all sets which are measurable for every μ belonging to the cone $\mathfrak{M}^+$ of all positive Radon measures.

For $E \in \mathfrak{L}_1$, we define the *dual of the preceding capacity*

$$c_{\Phi, p}(E) = \sup_\mu ||\mu||_1,$$

where $\mu \in L_1^+(E)$ with

$$||\Phi*\mu||_{p'} \leq 1, \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 < p < \infty.$$

Such a measure μ is called a *test measure* for $c_{\Phi,p}(E)$. In the case $p = 1$, we define

$$c_\Phi E = c_{\Phi,1}(E) = \sup_\mu ||\mu||_1,$$

the supremum being taken over all $\mu \in L_1^+(E)$ such that

$$\Phi*\mu(x) \geq 1 \ \forall x \in R^n.$$

Such a μ is called a *test measure* for $c_\Phi E$.

ν is called a c_Φ-*capacitary distribution for* E if it is a test measure for c_Φ and $||\nu||_1 = c_\Phi(E)$.

Let us define for all $E \subset R^n$,

$$c_\Phi^*(E) = \inf_{G \supset E} c_\Phi(G),$$

where G are open sets.

For $E \in \mathfrak{L}_1$, define the *capacit*

$$\tilde{c}_{\Phi,p}(E) = \sup_\mu ||\mu||_1,$$

where μ varies over the set of all $\mu \in L_1^+(E)$ such that

$$\Phi*(\Phi*\mu)^{\frac{1}{p-1}}(x) \leq 1 \quad \forall x \in R^n.$$

Such a μ is called a *test measure* for $\tilde{c}_{\Phi,p}(E)$.

If C is a capacity and $\mathfrak{A}$ its domain, i.e. a class of subsets of R^n which contains the compact sets and is closed under countable union, C is called an *inner capacity* if

$$E \in \mathfrak{A} \Rightarrow C(E) = \sup_F C(F),$$

the supremum being taken over all compact sets $F \subset E$. C is called an *outer capacity* if

$$E \in \mathfrak{A} \Rightarrow C(E) = \inf_G C(G),$$

the infimum being taken over all open $G \supset E$.

Proposition 1. $C_{\Phi,p}$ is an outer capacity.

Proposition 2. $c_{\Phi,p}$ is an inner capacity.

Proposition 3. We have

(i) $c_{\Phi,p}^*(E) = [C_{\Phi,p}(E)]^{\frac{1}{p}}$ $\forall E.$

(ii) $c_{\Phi,p}(E) = [C_{\Phi,p}(E)]^{\frac{1}{p}}$ $\forall$ *analytic set* E.

(For the proof of these 3 propositions, see N. MEYERS [12].)

Corollary. $c_{\Phi,p}(E) \leq [C_{\Phi,p}(E)]^{\frac{1}{p}}$ $\forall E \subset R^n$.

Indeed, from the preceding 2 propositions, we deduce that

$$c_{\Phi,p}(E) = \sup_F c_{\Phi,p}(F) \leq \inf_{G \supset E} c_{\Phi,p}(G) = c_{\Phi,p}^*(E) = [C_{\Phi,p}(E)]^{\frac{1}{p}},$$

where F are compact and G are open.

Now, in order to obtain Bessel capacities, we shall consider, in the definitions of the capacities from above, the particular kernel $\Phi(x) = g_\alpha(x)$ defined as

$$g_\alpha(x) = \frac{1}{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty e^{-t} e^{-\frac{|x|^2}{4t}} t^{\frac{\alpha-n}{2}} \frac{dt}{t}.$$

The kernel $g_\alpha(r) > 0$ is a strictly decreasing function of $r = |x|$, continuous outside the origin, $g_\alpha \in L^1(R^n)$ and as $x \to 0$,

$$g_\alpha(x) \sim |x|^{\alpha-n}, \quad 0 < \alpha < n,$$

$$g_n(x) \sim \log \frac{1}{|x|},$$

while, as $x \to \infty$,

$$g_\alpha(x) \sim |x|^{\frac{\alpha-n-1}{2}} e^{-|x|}, \quad 0 < \alpha \leq n.$$

For $\alpha, \beta > 0$, we have also the relation

(3) $$g_\alpha * g_\beta = g_{\alpha+\beta}$$

(cf. for instance N. MEYERS [12]).

And now, one introduces the 3 kinds of *Bessel capacities*

$$B_{\alpha,p} = C_{g_\alpha,p}, \quad b_{\alpha,p} = c_{g_\alpha,p}, \quad \tilde{b}_{\alpha,p} = \tilde{c}_{g_\alpha,p}.$$

Proposition 4. *Cap $E = 0$ iff (i.e. if and only if) there exists a function $f \in L_p^+$ such that the integral*

$$\int \frac{f(y)dy}{|x-y|^{n-1}} = \infty \quad \forall x \in E,$$

without being identically infinite.

(For the proof, see JU. G. REŠETNJAK [16], S. P. PREOBRAŽENSKIĬ [15], H. WALLIN [22, 23], or Y. MIZUTA [13].)

Corollary. Cap $E = 0 \Leftrightarrow B_{1,n}(E) = 0$.

In order to improve some of D. ADAMS and N. MEYERS [2] results, we have to introduce the following kernels:

$$g_{\alpha, p-1, m, \beta}(r) = \begin{cases} g_\alpha(r) \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{p-1} \left(\log_m \frac{1}{r}\right)^{\beta} & \text{for } r \leq r_m, \\ g_\alpha(r) \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r_m}\right)^{p-1} \left(\log_m \frac{1}{r_m}\right)^{\beta} & \text{for } r > r_m, \end{cases} \quad 0 < \alpha < n,$$

and

$$g_{n, p-1, m, \beta}(r) = \begin{cases} g_n(r) \left(\log \frac{1}{r}\right)^{p-2} \prod_{k=2}^{m-1} \left(\log_k \frac{1}{r}\right)^{p-1} \left(\log_m \frac{1}{r}\right)^{\beta} & \text{for } r \leq r_m, \\ g_n(r) \left(\log \frac{1}{r_m}\right)^{p-2} \prod_{k=2}^{m-1} \left(\log_k \frac{1}{r_m}\right)^{p-1} \left(\log_m \frac{1}{r_m}\right)^{\beta} & \text{for } r > r_m, \end{cases}$$

where r_m is here as well as everwhere in the paper such that $\log_m \frac{1}{r_m} > 1$. It is easy to see that $g_{\alpha, p-1, m, \beta}$ for $\alpha \leq n$ are kernels (according to the above definition). The corresponding capacities will be

$$c_{g_{\alpha, p-1, m, \beta}} = b_{(\alpha, p-1, m, \beta)}, \quad c^*_{g_{\alpha, p-1, m, \beta}} = B_{(\alpha, p-1, m, \beta)}, \quad 0 < \alpha \leq n.$$

L e m m a 1. *If* $\mu \in \mathfrak{M}^+$, $\alpha p \leq n$ *and* $\beta > p - 1$, *then*

$$\|g_\alpha * \mu\|_{p'} \leq Q \|g_{\alpha p, p-1, m, \beta} * \mu\|^{\frac{1}{p}} \|\mu\|^{\frac{1}{p'}} \quad (m = 1, 2, \ldots);$$

Q is a constant independent of μ *and* $\frac{1}{p} + \frac{1}{p'} = 1$

Holder inequality yields

$$g_\alpha * \mu(x) = (g_\alpha \, g_{\alpha p, p-1, m, \beta}^{-\frac{1}{p}} \cdot g_{\alpha p, p-1, m, \beta}^{\frac{1}{p}}) * \mu(x) \leq$$

$$\leq [(g_\alpha^{p'} \cdot g_{\alpha p, 1-p, m, \beta}^{-\frac{p'}{p}}) * \mu(x)]^{\frac{1}{p'}} [g_{\alpha p, p-1, m, \beta} * \mu(x)]^{\frac{1}{p}}.$$

Thus

$$\|g_\alpha * \mu\|_{p'} \leq [\int g_\alpha^{p'}(x) g_{\alpha p, p-1, m, \beta}^{-\frac{p'}{p}}(x) d(x)]^{\frac{1}{p'}} \|g_{\alpha p, p-1, m, \beta} * \mu\|_\infty^{\frac{1}{p}} \|\mu\|^{\frac{1}{p'}}.$$

It remains to show that

$$\int g_\alpha^{p'}(x) g_{\alpha p, p-1, m, \beta}^{-\frac{p'}{p}}(x) dx < \infty.$$

To do this, we need to investigate the behavior of the integrant only at $x = 0$ and $x = \infty$. We have

$$g_\alpha^{p'}(x) g_{\alpha p, p-1, m, \beta}^{-\frac{p'}{p}}(x) \sim |x|^{-\frac{n+1}{2}} e^{-|x|} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{|x|}\right)^{-1} \left(\log_m \frac{1}{|x|}\right)^{-\frac{\beta}{p-1}} \text{ as } x \to \infty,$$

and

$$g_\alpha^{p'}(x) g_{\alpha p, p-1, m, \beta}^{-\frac{p'}{p}}(x) \sim |x|^{-n} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{|x|}\right)^{-1} \left(\log_m \frac{1}{|x|}\right)^{-\frac{\beta}{p-1}} \text{ as } x \to 0.$$

It is easy to see that these 2 relations imply the preceding inequality, as desired.

Remark. This lemma generalizes a result of du PLESSIS [14], as well as lemma 3.1 of D. ADAMS and N. MEYERS [2].

P r o p o s i t i o n 5. *If* $f \in L^p$, *then, for* $0 < \alpha < n$, $2 < p < \infty$,

$$f_{\frac{\alpha}{p}}(x) = \int \frac{f(y) dy}{|x - y|^{n - \frac{\alpha}{p}}} < \infty$$

everywhere except possible in a set E which is of β*-capacity* $C_\beta E = 0 \; \forall \beta > n - \alpha$, *where* $C_\beta = C_\Phi$ *with* $\Phi(r) = r^{-\beta}$.

(For the proof, see N. du Plessis [14], theorem 4).

As a direct consequence of the preceding lemma, we have the following generalization of the preceding proposition:

C o r o l l a r y. *If* $f \in L^p$, *then, for* $0 < \alpha p < n$, $2 < p < \infty$, *we have* $g_\alpha * f(x) < \infty$ *everywhere except possible in a set E with* $B_{(\alpha p, p-1, m, \beta)}(E) = 0 \; \forall \beta > p - 1$ *and* $m = 1, 2, \ldots$.

Indeed, suppose, to prove it is false, that $g_\alpha * f(x) = \infty$ in a bounded set E with $B_{(\alpha p, p-1, m, \beta)}(E) > 0$ and let $\mu > 0$ be a measure with $\mu(R^n) = 1$, $S_\mu \subset E$ and such that $\int g_{\alpha p, p-1, m, \beta}(x - y) d\mu(y)$ is bounded in R^n. Then, arguing as in the preceding proposition and by means of the preceding lemma, it follows that

$$\int g_\alpha * f(x) d\mu(x) \leq \|f\|_p \|g_\alpha * \mu\|_{p'} \leq \|f\|_p Q \|g_{\alpha p, p-1, m, \beta} * \mu\|^{\frac{1}{p}} < \infty,$$

contradicting the hypotheses.

Remark. Du Plessis' proof of his theorem 4 (corresponding to our preceding proposition) is incorrect. He asserts that $\{\int [\int |x - y|^{\frac{\gamma}{r} - n} \times d\mu(y)]^{r' - \varepsilon} dx\}^{\frac{1}{r' - \varepsilon}} < \infty$ as a consequence of his lemma (generalized by our lemma 1), but the corresponding expression involved in his lemma is $\int_S [\int |x - y|^{\frac{\gamma}{r} - n} d\mu(y)]^{r' - \varepsilon} dx$, where S is supposed to be a bounded set, and this fact is essential in the proof.

THEOREM 1. *Let* $1 < q < q < \infty$; $\alpha p = n$ *and* $\beta > p - 1$, *then,* $\forall E \subset R^n$,

$$B_{(n,q)}(E) \leqq B_{(n,p-1,m,\beta)}(E) \leqq Q B_{\alpha,p}(E) \qquad (m = 2, 3, \ldots),$$

where Q *is a constant independent of* E *and the capacity* $B_{(n,q)}$ *corresponds to the kernel*

$$g_{n,q}(r) = \begin{cases} g_n(r)\left(\log \dfrac{1}{r}\right)^{q-2} & \text{for } r \leqq r_1, \\ g_n(r)\left(\log \dfrac{1}{r_1}\right)^{q-2} & \text{for } r > r_1, \end{cases}$$

with $\log \frac{1}{r_1} > 1$. *We have also*

$$B_{(n,p-1,m_1,\beta_1)}(E) \leqq B_{(n,p-1,m_2,\beta_2)}(E) \tag{4}$$

if $m_1 < m_2$ *and, for* $m_1 = m_2$, *if* $\beta_1 \geqq \beta_2$.

First, let F be a compact set with $b_{(n,p-1,m,\beta)}(F) > 0$. If μ is a non-zero test measure for $b_{(n,p-1,m,\beta)}(F)$, then, from the preceding lemma

$$\|g_\alpha * \mu\|_{p'} \leqq Q^{\frac{1}{p}} \|\mu\|^{\frac{1}{p'}},$$

for some constant Q independent of F. Hence, $\nu = \dfrac{\mu}{Q^{\frac{1}{p}} \|\mu\|^{\frac{1}{p'}}}$ is a test measure for $b_{\alpha,p}(F)$, so that

$$\|\nu\|_1 \leqq b_{\alpha,p}(F).$$

Thus, on account of proposition 3,

$$b_{(n,p-1,m,\beta)}(F) \leqq Q B_{\alpha,p}(F),$$

which is clearly true also for $b_{(n,p-1,m,\beta)}(F) = 0$, and since $b_{(n,p-1,m,\beta)}$ is an inner capacity and $B_{\alpha,p}$ an outer capacity, it follows that

$$B_{(n,p-1,m,\beta)}(E) \equiv b^*_{(n,p-1,m,\beta)}(E) \leqq Q B_{\alpha,p}(E).$$

The inequality

$$B_{(n,q)}(E) \leqq B_{(n,p-1,m,\beta)}(E)$$

is a consequence of the fact that

$$\lim_{r \to 0} \frac{g_{n,p-1,m,\beta}(r)}{g_{n,q}(r)} = 0.$$

Finally, inequality (4) follows from

$$\lim_{r \to 0} \frac{g_{n,p-1,m_2,\beta_2}(r)}{g_{n,p-1,m_1,\beta_1}(r)} = 0.$$

Remark. This theorem generalizes D. ADAMS and N. MEYERS theorem 3.1 in [2].

From the preceding theorem and the corollary of proposition 4, we deduce the

Corollary. *In the hypotheses of the preceding theorem, we have the inequalities*

$$B_{(n,q)}(E) \leqq B_{(n,n-1,m,\beta)}(E) \leqq Q \operatorname{cap} E \quad (m = 1, 2, \ldots).$$

Proposition 6. $\forall E \subset R^n$, $C_\Phi(E) > 0$ *or* $c^*_\Phi(E) > 0 \Rightarrow H_{\frac{1}{\Phi}}(E) = \infty$ *and if*

$$\int_0^{r_0} \Phi(r)\, dh(r) < \infty, \text{ \it then } C_\Phi(E) = 0 \text{ \it or } c^*_\Phi(E) = 0 < H_{\frac{1}{\Phi}}(E) = 0.$$

(The proof in the case C_Φ is given by S. J. Taylor [19] theorems 1 and 2 while, for the case c^*_Φ, we may use the methods of chap. IV in Carleson's book [8] and the capacitability results of B. Fuglede [10]).

Theorem 2. *Let* $1 < q < p < \infty$ *and* $\beta < 1$, *then*

$$B_{\alpha,p} \preccurlyeq H_h \preccurlyeq B_{(n,p,-1,m,\beta)} \preccurlyeq B_{(n,q)} \quad (m = 1, 2, \ldots),$$

where $h(r) = \left(\log \frac{1}{r}\right)^{1-p}$ *and* $B_{(n,p,-1,m,\beta)}$ *corresponds to the kernel*

$$g_{n,p,-1,m,\beta}(r) = \begin{cases} g_n(r)\left(\log \dfrac{1}{r}\right)^{p-2} \displaystyle\prod_{k=2}^{m-1}\left(\log_k \frac{1}{r}\right)^{-1}\left(\log_m \frac{1}{r}\right)^{-\beta} & \text{for } r \leqq r_m, \\ g_n(r)\left(\log \dfrac{1}{r_m}\right)^{p-2} \displaystyle\prod_{k=2}^{m-1}\left(\log_k \frac{1}{r_m}\right)^{-1}\left(\log_m \frac{1}{r_m}\right)^{-\beta} & \text{for } r > r_m. \end{cases}$$

We have also $B_{(n,p,-1,m_1,\beta_1)} \prec B_{(n,p,-1,m_2,\beta_2)}$ *if* $m_1 < m_2$ *and for* $m_1 = m_2$ *if* $\beta_1 < \beta_2$.

We recall that if C and C' are 2 capacities with the same domain, C' is said to be weaker than C (or C is stronger than C') and write $C' \preccurlyeq C$ (or $C \succcurlyeq C'$) if $C(E) = 0 \Rightarrow C'(E) = 0$. We say that C and C' are equivalent and write $C \sim C'$ if $C \preccurlyeq C'$ and $C' \preccurlyeq C$. Finally, if $C' \preccurlyeq C$ but C and C' are not equivalent, we say that C' is strictly weaker than C (or C is strictly stronger than C') and write $C' \prec C$ (or $C \succ C'$).

The inclusion $B_{\alpha,p} \preccurlyeq H_h$ was established by D. ADAMS and N. MEYERS [2], while $B_{(n,p,-1,m,\beta)} \preccurlyeq B_{(n,q)}$ is obvious. Now, let us establish also that $H_h \preccurlyeq B_{(n,p,-1,m,\beta)}$.

Indeed, on account of the preceding proposition,

$$\int_0^{r_m} g_{n,p,-1,\beta}(r)\,d\left[\left(\log\frac{1}{r}\right)^{1-p}\right] \leqq \text{const.} \int_{r_m}^{0} \prod_{k=1}^{m-1}\left(\log_k\frac{1}{r}\right)^{-1}\left(\log_m\frac{1}{r}\right)^{-\beta} d\left(\log\frac{1}{r}\right) < \infty$$

implies the desired inclusion.

Remark. This is an extension of theorem 3.2 in D. ADAMS and N. MEYERS paper [2].

L e m m a 2. *If* $\alpha p \leqq n$, *then*

$$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x) = g_{\alpha(p-1),p-2,m,\beta} * g_\alpha(x) \leqq g_{\alpha p,p-2,m,\beta}(x),$$

and if $1 < p \leqq 2$, *and, in the particular cases* $p = 2$ *or* $m = 1$, *if* $\beta \leqslant 0$, *then* const. $g_{\alpha p,p-2,m,\beta}(x) \leqq g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x) = g_{\alpha(p-1),p-2,m,\beta} * g_\alpha(x)$, *where* $g_{\alpha,p-2,m,\beta}$ *is obtained from the expression of* $g_{\alpha,p-1,m,\beta}$ *by taking* $p-2$ *instead of* $p-1$.

In order to prove the relation

$$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x) = g_{\alpha(p-1),p-2,m,\beta} * g_\alpha(x),$$

we observe that

$$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x) = \int g_\alpha(x-y)\,g_{\alpha(p-1),p-2,m,\beta}(y)dy =$$

$$= \int g_{\alpha(p-1),p-2,m,\beta}(x-z)g_\alpha(z)dz = g_{\alpha(p-1),p-2,m,\alpha} * g_\alpha(x).$$

where we used the transformation $x - y = z$, hence $y = x - z$.

For the remaining 2 inequalities, clearly, we need only to verify them as $x \to 0$ and $x \to \infty$.

I. Suppose first $p \geqq 2$ and, if $p = 2$ or $m = 1$, assume also $\beta \geqq 0$.

I.1. *The case* $x \to 0$. Let $0 < 2\rho = |x| \leqq 2r_m$; then

$$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x) = \int_{|y|<\frac{|x|}{2}} g_\alpha(x-y)g_{\alpha(p-1),p-2,m,\beta}(y)dy +$$

$$+ \int_{|x-y|<\frac{|x|}{2}} g_\alpha(x-y)g_{\alpha(p-1),p-2,m,\beta}(y)dy +$$

$$\int_{\{|y|\geqq\frac{|x|}{2},\,|x-y|\geqq\frac{|x|}{2}\}} g_\alpha(x-y)g_{\alpha(p-1),p-2,m,\beta}(y)dy = I_1 + I_2 + I_3.$$

(5) $$I_1 \leqq \text{const.}\ g_\alpha(\rho)\int_0^\rho |y|^{\alpha(p-1)}\prod_{k=1}^{m-1}\left(\log_k\frac{1}{|y|}\right)^{p-2}\left(\log_m\frac{1}{|y|}\right)^\beta \frac{d|y|}{|y|} \leqq$$

$$\leqq \text{const.}\ g_\alpha(\rho)\rho^{\frac{\alpha(p-1)}{2}}\prod_{k=1}^{m-1}\left(\log_k\frac{1}{\rho}\right)^{p-2}\left(\log_m\frac{1}{\rho}\right)^\beta\int_0^\rho r^{\frac{\alpha(p-1)}{2}-1}dr =$$

$$= \text{const.}\ g_\alpha(\rho)\ \rho^{\alpha(p-1)}\prod_{k=1}^{m-1}\left(\log_k\frac{2}{|x|}\right)^{p-2}\left(\log_m\frac{2}{|x|}\right)^\beta \leqq$$

$$\leqq \text{const.}\ \rho^{\alpha p-n}\prod_{k=1}^{m-1}\left[\log_{k-1}\left(\log 2+\log\frac{1}{|x|}\right)\right]^{p-2}\left[\log_{m-1}\left(\log 2+\log\frac{1}{|x|}\right)\right]^\beta \leqq$$

$$\leqq \text{const.}\ g_{\alpha p}(\rho)\prod_{k=1}^{m-1}\left[\log_{k-1}\left(2\log\frac{1}{|x|}\right)\right]^{p-2}\left[\log_{m-1}\left(2\log\frac{1}{|x|}\right)\right]^\beta \leqq$$

$$\leqq \text{const.}\ g_{\alpha p}(x)\prod_{k=1}^{m-1}\left(\log_k\frac{1}{|x|}\right)^{p-2}\left(\log_m\frac{1}{|x|}\right)^\beta = \text{const.}\ g_{\alpha p,p-2,m,\beta}(x).$$

Next, on account of (3) and arguing as above,

(6) $$I_2 \leqq \prod_{k=1}^{m-1}\left(\log_k\frac{1}{\rho}\right)^{p-2}\left(\log_m\frac{1}{\rho}\right)^\beta\int_{|x-y|<\rho} g_\alpha(x-y)g_{\alpha(p-1)}(y)\,dy \leqq$$

$$\leqq \text{const.}\prod_{k=1}^{m-1}\left(\log_k\frac{1}{|x|}\right)^{p-2}\left(\log_m\frac{1}{|x|}\right)^\beta g_\alpha * g_{\alpha(p-1)}(x) = \text{const.}\ g_{\alpha p,p-2,m,\beta}(x),$$

(7) $$I_3 \leqq \prod_{k=1}^{m-1}\left(\log_k\frac{1}{\rho}\right)^{p-2}\left(\log_m\frac{1}{\rho}\right)^\beta\int_{\{|y|\geqq\rho,\,|x-y|\geqq\rho\}} g_\alpha(x-y)g_{\alpha(p-1)}(y)dy \leqq$$

$$\leqq \text{const.}\prod_{k=1}^{m-1}\left(\log_k\frac{1}{|x|}\right)^{p-2}\left(\log_m\frac{1}{|x|}\right)^\beta g_{\alpha p}(x) = \text{const.}\ g_{\alpha p,p-2,m,\beta}(x).$$

Combining (5), (6), (7) yields

(8) $$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x) \leqq \text{const.}\ g_{\alpha p,p-2,m,\beta}(x)\ \text{ for } |x| \leqq 2r_m$$

2. *The case* $x \to \infty$.

$$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x) = \int_{|y|\leqq r_m} g_\alpha(x-y)g_{\alpha(p-1)}(y)\prod_{k=1}^{m-1}\left(\log_k\frac{1}{|y|}\right)^{p-2}\left(\log_m\frac{1}{|y|}\right)^\beta\cdot dy +$$

$$+ \int_{|y|>r_m} g_\alpha(x-y)g_{\alpha(p-1),p-2,m,\beta}(y)\prod_{k=1}^{m-1}\left(\log_k\frac{1}{|y|}\right)^{p-2}\left(\log_m\frac{1}{|y|}\right)^\beta dy = I_4 + I_5.$$

If $|x| > 1$, then

$$I_4 \leqq \text{const. } g_\alpha(|x| - r_m) \int_0^{r_m} |y|^{\alpha(p-1)-1} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{|y|}\right)^{p-2} \left(\log_m \frac{1}{|y|}\right)^{\beta} d|y| \leqq$$

$$\leqq \text{const. } g_\alpha\left(\frac{x}{2}\right) r_m^{\frac{\alpha(p-1)}{2}} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r_m}\right)^{p-2} \left(\log_m \frac{1}{r_m}\right)^{\beta} \int_0^{r_m} r^{\frac{\alpha(p-1)}{2}-1} dr \leqq$$

$$\leqq \text{const. } g_{\alpha, p-2, m, \beta}(x) \leqq \text{const. } g_{\alpha p, p-2, m, \beta}(x)$$

and, taking into account also (3),

$$I_5 = \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r_m}\right)^{p-2} \left(\log_m \frac{1}{r_m}\right)^{\beta} \int_{|y| > r_m} g_\alpha(x-y) g_{\alpha(p-1)}(y) dy \leqq g_{\alpha p, p-2, m, \beta}(x).$$

Thus, $g_\alpha * g_{\alpha(p-1), p-2, m, \beta}(x) \leqq \text{const. } g_{\alpha p, p-2, m, \beta}(x)$ for $|x| > 1$, hence and on account of (8), we are allowed to conclude that $g_\alpha * g_{\alpha(p-1), p-2, m, \beta}(x) \leqq \text{const. } g_{\alpha p, p-2, m, \beta}(x)$ everywhere in the case $p \geqq 2$ and, if $p = 2$ or $m = 1$, for $\beta \geqq 0$.

II. Now, let us suppose $1 < p < 2$ or $(p = 2, \beta \leqq 0)$, or $(m = 1, \beta \leqq 0)$.

II.1. *The case* $x \to 0$.

$$I_1 \leqq \prod_{k=1}^{m-1} \left(\log_k \frac{1}{\rho}\right)^{p-2} \left(\log_m \frac{1}{\rho}\right)^{\beta} \int_{|y| < \rho} g_\alpha(x-y) g_{\alpha(p-1)}(y) dy \leqq g_{\alpha p, p-2, m, \beta}(x).$$

$$I_2 \leqq \prod_{k=1}^{m-1} \left(\log_k \frac{1}{3\rho}\right)^{p-2} \left(\log_m \frac{1}{3\rho}\right)^{\beta} \int_{|x-y| < \rho} g_\alpha(x-y) g_{\alpha(p-1)}(y) dy \leqq$$

$$\prod_{k=1}^{m-1} \left[\log_{k-1} \left(\log \frac{1}{\rho} - \log 3\right)\right]^{p-2} \left[\log_{m-1} \left(\log \frac{1}{\rho} - \log 3\right)\right]^{\beta} g_{\alpha p}(x) \leqq$$

$$\prod_{k=1}^{m-1} \left[\log_{k-1} \left(\frac{1}{2} \log \frac{1}{\rho}\right)\right]^{p-2} \left[\log_{m-1} \left(\frac{1}{2} \log \frac{1}{\rho}\right)\right]^{\beta} g_{\alpha p}(x) \leqq$$

$$\text{const. } \prod_{k=1}^{m-1} \left(\log_k \frac{1}{\rho}\right)^{p-2} \left(\log_m \frac{1}{\rho}\right)^{\beta} g_{\alpha p}(x) \leqq \text{const. } g_{\alpha p, p-2, m, \beta}(x).$$

$$I_3 \leqq \prod_{k=1}^{m-1} \left(\log_k \frac{1}{\rho}\right)^{p-2} \left(\log_m \frac{1}{\rho}\right)^{\beta} \rho^{-\delta} \int_{\{|y| \geqq \rho,\ |x-y| \geqq \rho\}} g_\alpha(x-y) g_{\alpha(p-1)}(y) |y|^{\delta} dy \leqq$$

$$\leqq \text{const. } \prod_{k=1}^{m-1} \left(\log_k \frac{1}{\rho}\right)^{p-2} \left(\log_m \frac{1}{\rho}\right)^{\beta} \rho^{-\delta} \int_{\{|y| \geqq \rho,\ |x-y| \geqq \rho\}} g_\alpha(x-y) g_{\alpha(p-1)+\delta}(y) dy \leqq$$

$$\leqq \text{const. } \prod_{k=1}^{m-1} \left(\log_k \frac{1}{|x|}\right)^{p-2} \left(\log_m \frac{1}{|x|}\right)^{\beta} g_{\alpha p+\delta}(x) |x|^{-\delta} \leqq \text{const. } g_{\alpha p, p-2, m, \beta}(x).$$

2. *The case* $x \to \infty$.

$$I_4 \leqq \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r_m}\right)^{p-2} \left(\log_m \frac{1}{r_m}\right)^{\beta} g_\alpha * g_{\alpha(p-1)}(x) = g_{\alpha p, p-2, m, \beta}(x).$$

$$I_5 \leqq \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r_m}\right)^{p-2} \left(\log_m \frac{1}{r_m}\right)^{\beta} g_{\alpha p}(x) = g_{\alpha p, p-2, m, \beta}(x).$$

Thus, also in the new hypotheses: $p < 2$ or $(p = 2, \beta < 0)$, or $(m = 1, \beta < 0)$, we have the same majoration for $I_1 - I_5$, allowing us to conclude that the inequality $g_\alpha * g_{\alpha(p-1), p-2, m, \beta}(x) \leqq \text{const. } g_{\alpha p, p-2, m, \beta}(x)$ for $p > 1$ and any real β.

II'. Now, let us prove the opposite inequality if $1 < p < 2$ or $(p = 2, \beta < 0)$, or $(m = 1, \beta < 0)$.

II'.1. *The case* $x \to 0$. Suppose first $\alpha p = n$.

$$g_\alpha * g_{\alpha(p-1), p-2, m, \beta}(x) \geqq I_3 \geqq \prod_{k=1}^{m-1} \left(\log_k \frac{1}{\rho}\right)^{p-2} \left(\log_m \frac{1}{\rho}\right)^{\beta} [g_\alpha * g_{\alpha(p-1)}(x) -$$

$$- \int_{|y|<\rho} g_\alpha(x-y) g_{\alpha(p-1)}(y) dy - \int_{|x-y|<\rho} g_\alpha(x-y) g_{\alpha(p-1)}(y) dy\Big] \geqq$$

$$\geqq \prod_{k=1}^{m-1} \left(\log_k \frac{2}{|x|}\right)^{p-2} \left(\log_m \frac{2}{|x|}\right)^{\beta} \left[g_{\alpha p}(x) - \text{const. } g_\alpha(\rho) \int_0^{\rho} r^{n-\alpha-1} dr -\right.$$

$$\left. - \text{const. } g_{n-\alpha}(\rho) \int_0^{\rho} r^{\alpha-1} dr\right] \geqq \prod_{k=1}^{m-1} \left[\log_{k-1} \left(\log 2 + \log \frac{1}{|x|}\right)\right]^{p-2} \left[\log_{m-1} \left(\log 2 +\right.\right.$$

$$\left.\left. + \log \frac{1}{|x|}\right)\right]^{\beta} [g_n(x) - \text{const. } g_\alpha(\rho)\rho^{n-\alpha} - \text{const. } g_{n-\alpha}(\rho)\rho^{\alpha-1}] \geqq$$

$$\geqq \prod_{k=1}^{m-1} \left[\log_{k-1} \left(2 \log \frac{1}{|x|}\right)\right]^{p-2} \left[\log_{m-1} \left(2 \log \frac{1}{|x|}\right)\right]^{\beta} [g_n(x) - \text{const.}] \geqq$$

$$\geqq \text{const. } \prod_{k=1}^{m-1} \left(\log_k \frac{1}{|x|}\right)^{p-2} \left(\log_m \frac{1}{|x|}\right)^{\beta} \left[g_n(x) - \frac{1}{2} g_n(x)\right] \geqq \text{const. } g_{n, p-2, m, \beta}(x).$$

For $\alpha p < n$, we have $g_\alpha * g_{\alpha(p-1), p-2, m, \beta}(x) \geqq$

$$\geqq I_1 \geqq g_\alpha(3\rho) \prod_{k=1}^{m-1} \left(\log_k \frac{1}{\rho}\right)^{p-2} \left(\log_m \frac{1}{\rho}\right)^{\beta} \rho^{-\delta} \int_{|y|<\rho} g_{\alpha(p-1)}(y) |y|^{\delta} dy \geqq$$

$$\geqq \text{const. } g_\alpha(\rho)\prod_{k=1}^{m-1}\left(\log_k\frac{2}{|x|}\right)^{p-2}\left(\log_m\frac{2}{|x|}\right)^{\beta}\rho^{-\delta}\int_0^{\rho} r^{\alpha(p-1)+\delta-1}\,dr \geqq$$

$$\geqq \text{const. } g_\alpha(\rho)\prod_{k=1}^{m-1}\left(\log_k\frac{1}{|x|}\right)^{p-2}\left(\log_m\frac{1}{|x|}\right)^{\beta}\rho^{\alpha(p-1)}\geqq$$

$$\geqq \text{const. } g_{\alpha p}(\rho)\prod_{k=1}^{m-1}\left(\log_k\frac{1}{|x|}\right)^{p-2}\left(\log_m\frac{1}{|x|}\right)^{\beta}\geqq \text{const. } g_{\alpha p,p-2,m,\beta}(x).$$

2. *The case* $x\to\infty$.

$$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x) \geqq I_4 \geqq$$

$$\geqslant \text{const.}\prod_{k=1}^{m-1}\left(\log_k\frac{1}{r_m}\right)^{p-2}\left(\log_m\frac{1}{r_m}\right)^{\beta} g_\alpha(3r_m)\int_0^{r_m} r^{\alpha(p-1)-1}\,dr\geqq$$

$$\geqq \text{const.}\prod_{k=1}^{m-1}\left(\log_k\frac{1}{r_m}\right)^{p-2}\left(\log_m\frac{1}{r_m}\right)^{\beta} g_\alpha(r_m)\, r_m^{\alpha(p-1)}\geqq$$

$$\geqq \text{const.}\prod_{k=1}^{m-1}\left(\log_k\frac{1}{r_m}\right)^{p-2}\left(\log_m\frac{1}{r_m}\right)^{\beta} g_{\alpha p}(r_m) \geqq \text{const. } g_{\alpha p,p-2,m,\beta}(x).$$

Thus

$$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x)\geqq \text{const. } g_{\alpha p,p-2,m,\beta}(x)\ \text{ for } |x|>1,$$

and this completes the proof of our lemma.

Under the more restrictive condition $\alpha p<n$, I obtained the following stronger result generalizing lemma 3.2. of D. ADAMS and N. MEYERS paper [2]:

Lemma 3. *If* $\alpha p<n$, *then*

$$g_{\alpha(p-1),p-2,m,\beta} * g_\alpha = g_\alpha * g_{\alpha(p-1),p-2,m,\beta}\sim g_{\alpha p,p-2,m,\beta}.$$

On account of the preceding lemma, we have only to prove that

$$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x)\geqq \text{const. } g_{\alpha p,p-2,m,\beta}(x)$$

also for $2<p<\infty$.

Using the notation of the preceding lemma, we get

1. *The case* $x\to 0$.

$$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x)\geqq I_1\geqq$$

$$\geqq \text{const.}\prod_{k=1}^{m-1}\left(\log_k\frac{1}{\rho}\right)^{p-2}\left(\log_m\frac{1}{\rho}\right)^{\beta} g_\alpha(3\rho)\int_0^\rho r^{\alpha(p-1)-1}\,dr =$$

$$=\text{const.}\prod_{k=1}^{m-1}\left(\log_k\frac{1}{\rho}\right)^{p-2}\left(\log_m\frac{1}{\rho}\right)^{\beta}\rho^{\alpha p-n}\geqq \text{const. } g_{\alpha p,p-2,m,\beta}(\rho)\geqq$$

$$\geqq \text{const. } g_{\alpha p,p-2,m,\beta}(x).$$

2. *The case* $x\to\infty$.

$$g_\alpha * g_{\alpha(p-1),p-2,m,\beta}(x)\geqslant I_5 =$$

$$=\prod_{k=1}^{m-1}\left(\log_k\frac{1}{r_m}\right)^{p-2}\left(\log_m\frac{1}{r_m}\right)^{\beta}\int_{|y|>r_m} g_\alpha(x-y)\,g_{\alpha(p-1)}(y)dy\geqq$$

$$\geqq\prod_{k=1}^{m-1}\left(\log_k\frac{1}{r_m}\right)^{p-2}\left(\log_m\frac{1}{r_m}\right)^{\beta}\Big[g_\alpha * g_{\alpha(p-1)}(x) - \int_{|y|\leqslant r_m} g_\alpha(x-y)\,g_{\alpha(p-1)}(y)dy\Big]\geqq$$

$$\geqq\prod_{k=1}^{m-1}\left(\log_k\frac{1}{r_m}\right)^{p-2}\left(\log_m\frac{1}{r_m}\right)^{\beta}\big[g_{\alpha p}(x) - \text{const. } g_\alpha(|x|-r_m)\, r_m^{\alpha(p-1)}\geqq[$$

$$\geqq\prod_{k=1}^{m-1}\left(\log_k\frac{1}{r_m}\right)^{p-2}\left(\log_m\frac{1}{r_m}\right)^{\beta}\left[g_{\alpha p}(x)-\text{const. } g_\alpha\left(\frac{x}{2}\right)\right]\geqq$$

$$\geqq\prod_{k=1}^{m-1}\left(\log_\alpha\frac{1}{r_m}\right)^{p-2}\left(\log_m\frac{1}{r_m}\right)^{\beta}\left[g_{\alpha p}(x)-\frac12 g_{\alpha p}(x)\right]\geqq\frac12 g_{\alpha p,p-2,m,\beta}(x),$$

as desired.

Lemma 4. *If* $\alpha p\leqq n$ *and* $\mu\in\mathfrak{M}^+$, *then* $\forall E\subset R^n$,

(9) $[g_\alpha*(g_\alpha*\mu)^{\frac{1}{p-1}}(x)]^{p-1}\leqq Q_1\, g_{\alpha p,p-2,m,\beta}*\mu(x),\ 2\leqq p<\infty,\ \beta>p-2,$

(10) $[g_\alpha*(g_\alpha*\mu)^{\frac{1}{p-1}}(x)]^{p-1}\geqq Q_2\, g_{\alpha p,p-2,m,\beta}*\mu(x),\ 1<p\leqslant 2,\ \beta<p-2;$

if $p=2$, *the inequalities* (9) *and* (10) *hold for* $\beta\geqq 0$ *and* $\beta\leqq 0$, *respectively;* $Q_1, Q_2<0$ *are constants independent of* μ.

First, consider the case $p>2$; by Hölder's inequality,

(11) $$g_\alpha*(g_\alpha*\mu)^{\frac{1}{p-1}}(x)=\int g_\alpha(x-y)\Big[\int g_\alpha(y-z)\,d\mu(z)\Big]^{\frac{1}{p-1}}dy=$$

$$=\int g_\alpha(x-y)\, g_{\alpha(p-1),p-2,m,\beta}^{\frac{-1}{p-1}}(x-y)\Big[\int g_\alpha(y-z)\, g_{\alpha(p-1),p-2,m,\beta}(x-y)d\mu(z)\Big]^{\frac{1}{p-1}}dy\leqq$$

$$\leqq\Big[\int g_\alpha^{\frac{p-1}{p-2}}(x-y)\, g_{\alpha(p-1),p-2,m,\beta}^{\frac{-1}{p-2}}(x-y)dy\Big]^{\frac{p-2}{p-1}}\Big[\int g_{\alpha(p-1),p-2,m,\beta}(x-y)\cdot\int g_\alpha(y-z)\,d\mu(z)\,dy\Big]^{\frac{1}{p-1}}=$$

$$=\Big[\int g_\alpha^{\frac{p-1}{p-2}}(y)\, g_{\alpha(p-1),p-2,m,\beta}^{\frac{-1}{p-2}}(x-y)dy\Big]^{\frac{p-2}{p-1}}\big[g_{\alpha(p-1),p-2,m,\beta}*g_\alpha*\mu(x)\big]^{\frac{1}{p-1}}.$$

But,

$$(12)\qquad \int_0^{\rho} g_\alpha^{\frac{p-1}{p-2}}(y)\, g_{\alpha(p-1),\,p-2,\,m,\,\beta}^{\frac{-1}{p-2}}(y)dy \leqq$$

$$\leqq \text{const.} \int_0^{\rho} r^{\frac{(\alpha-n)(p-1)-\alpha(p-1)+n}{p-2}+n-1} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{-1} \cdot \left(\log_m \frac{1}{r}\right)^{\frac{-\beta}{p-2}} dr =$$

$$= \text{const.}\; u^{1-\frac{\beta}{p-2}} \Big|_{\log_m \frac{1}{\rho}}^{\infty} = Q_1' < \infty$$

for $\beta > p-2$ and $\rho \leqslant r_m$. Next, for $R < r_m$,

$$\int_R^{\infty} g_\alpha^{\frac{p-1}{p-1}}(y)\, g_{\alpha(p-1),\,p-2,\,m,\,\beta}^{\frac{-1}{p-2}}(y)dy \leqslant \text{cnost.} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r_m}\right)^{-1} \cdot$$

$$\cdot \left(\log_m \frac{1}{r_m}\right)^{\frac{-\beta}{p-2}} \int_R^{\infty} e^{-2r}\, r^{n-2} dr = Q_2' < \infty,$$

hence and taking into account (11), (12) and lemma 2, we deduce (9), where $Q_1 = [\max\ (Q_1', Q_2')]^{p-1}$.

And now, consider the case $1 < p < 2$; from lemma 2 and Hölder inequality, we get

$$g_{\alpha p,\,p-2,\,m,\,\beta} * \mu(x) \leqq \text{const.}\; g_{\alpha(p-1),\,p-2,\,m,\,\beta} * g_\alpha * \mu(x) =$$

$$= \text{const.} \int g_{\alpha(p-1),\,p-2,\,m,\,\beta}(x-y)\, g_\alpha^{1-p}(x-y)\, g_\alpha^{p-1}(x-y) \int g_\alpha(y-z)d\mu(z)dy \leqq$$

$$\leqq \text{const.} \left[\int g_\alpha^{\frac{1-p}{2-p}}(x-y)\, g_{\alpha(p-1),\,p-2,\,m,\,\beta}^{\frac{1}{2-p}}(x-y)\, dy\right]^{2-p} \cdot$$

$$\cdot \left\{\int g_\alpha(x-y) \left[g_\alpha(y-z)d\mu(z)\right]^{\frac{1}{p-1}} dy\right\}^{p-1} =$$

$$= \text{const.} \left[\int g_\alpha^{\frac{1-p}{2-p}}(y)\, g_{\alpha(p-1),\,p-2,\,m,\,\beta}^{\frac{1}{2-p}}(y)\, dy\right]^{2-p} \left[g_\alpha * (g_\alpha * \mu)^{\frac{1}{p-1}}(x)\right]^{p-1},$$

and arguind as above, it follows that

$$\int g_\alpha^{\frac{1-p}{2-p}}(y)\, g_{\alpha(p-1),\,p-2,\,m,\,\beta}^{\frac{1}{2-p}}(y)\, dy < \infty,$$

hence, we obtain (10).

Finally, in the case $p = 2$, on account of (3), if $\beta \geqslant 0$:

$$g_\alpha * g_\alpha * \mu(x) = g_{2\alpha} * \mu(x) = g_{\alpha p} * \mu(x) \leqslant \int g_{\alpha p}(x-y) \left(\log_m \frac{1}{|x-y|}\right)^{\beta} d\mu(y) =$$

$$= g_{\alpha p,\,p-2,\,m,\,\beta} * \mu(x),$$

and if $\beta \leqslant 0$:

$$g_{\alpha p,\,p-2,\,m,\,\beta} * \mu(x) = \int g_{2\alpha}(x-y) \left(\log_m \frac{1}{|x-y|}\right)^{\beta} d\mu(y) \leqslant g_{2\alpha} * \mu(x) =$$

$$= g_\alpha * g_\alpha * \mu(x),$$

which completes the proof of our lemma.

Remark. This is an extension of lemma 3.3 of D. ADAMS and N. MEYERS [2].

P r o p o s i t i o n 7. *If $E \subset R^n$ is an analytic set, then*

$$\tilde{c}_{\Phi,p}(E) \leqslant C_{\Phi,p}(E) \leqslant Q\, \tilde{c}_{\Phi,p}(E).$$

If we replace $\tilde{c}_{\Phi,p}$ by $\tilde{c}^{*}_{\Phi,p}$, then the above inequality holds for all sets E.

THEOREM 3. *Let $\alpha p \leqslant n$, then*

$$(13)\qquad B_{(\alpha p,q)}(E) \leqslant B_{(\alpha p,p-2,m,\beta)}(E) \leqslant Q_1 B_{\alpha,p}(E) \qquad (2 \leqslant p < q < \infty,\ \beta > p-2),$$

$$B_{\alpha,p}(E) \leqslant Q_2 B_{(\alpha p,p-2,m,\beta)}(E) \leqslant Q_2\, B_{(\alpha p,q)}(E) \qquad (1 < q < p \leqslant 2,\ \beta < p-2);$$

if $p = 2$, the preceding double inequalities hold for $\beta \geqslant 0$ and $\beta \leqslant 0$, respectively, Q_1, Q_2 are constants independent of E and the capacity $B_{\alpha p,\,p-2,m,\beta)}$ corresponds to the kernel $g_{\alpha p,\,p-2,\,m,\,\beta}$.

We consider the first inequality. Let μ be a test measure for $b_{\alpha p,\,p-2,\,m,\,\beta)}(F)$, where F is supposed to be compact. Then, by the proceding lemma, $Q_1^{-1}\mu$ is a test measure for $\tilde{b}_{\alpha,p}(F)$. Hence and from the preceding proposition,

$$\|\mu\|_1 \leqslant Q_1 \tilde{b}_{\alpha,p}(F) \leqslant Q_1 B_{\alpha,p}(F).$$

Thus

$$b_{(\alpha p,\,p-2,\,m,\,\beta)}(F) \leqslant Q_1 \tilde{b}_{\alpha,p}(F) \leqslant Q_1 B_{\alpha,p}(F),$$

and since $b_{(\alpha p,\,p-2,\,m,\,\beta)}$ is an inner capacity, and $B_{\alpha,p}$ an outer capacity, it follows that

$$B_{(\alpha p,\,p-2,\,m,\,\beta)}(E) \equiv b^{*}_{(\alpha p,\,p-2,\,m,\,\beta)}(E) \leqslant Q_1 B_{\alpha,p}(E).$$

The inequality

$$B_{(\alpha p,q)}(E) \leqslant B_{(\alpha p,\,p-2,\,m,\,\beta)}(E)$$

is a direct consequence of the relation

$$\lim_{r\to 0} \frac{g_{\alpha p,q}(r)}{g_{\alpha p,\,p-2,\,m,\,\beta}(r)} = \infty.$$

The second inequality in this lemma may be proved in a similar way.

Remark. This result improves theorem 1, except for the case $1 < p < 2$ and represents an extension of D. ADAMS and N. MEYERS' theorem 3.3 in [2].

Corollary. Cap $E = 0 \Rightarrow B_{n,\, n-2,\, m,\, \beta}(E) = 0 \ \forall \beta > n-2 \ (m = 1, 2, \ldots)$.

Let us recall some notations. For $\mu \in \mathfrak{M}^+$, set

$$v(\mu, x, \rho) = \int_{B(x,\rho)} d\mu(y)$$

for $x \in R^n$, $0 \leqslant \rho < \infty$ and also

$$v(\mu, \rho) = \sup_{x \in R^n} v(\mu, x, \rho).$$

P r o p o s i t i o n 8. *Let $E \subset R^n$ be an analytic set. $H_h(E) > 0$ iff there exists $\mu \in \mathfrak{M}^+$, $\mu \neq 0$, such that*

$$v(\mu, \rho) \leqslant h(\rho),\ 0 < \rho \leqslant \rho_0.$$

(For a proof, see L. CARLESON [8], chap, II. theorem 1, p. 7.)

Arguing as in D. ADAMS and N. MEYERS' lemma 4.1 of [2], we obtain

L e m m a 5. *Let $\mu \in L_1^+$ and $\lambda \in \mathfrak{M}^+$; suppose that $\forall \rho$, $0 < \rho \leqslant \rho_0 < 1$,*

$$v(\mu, \rho) \leqslant Q_1 \rho^{d_1} \prod_{k=1}^{m} \left(\log_k \frac{1}{\rho}\right)^{-S_k}$$

and

$$v(\lambda, \rho) \leqslant Q_2 \rho^{d_2} \prod_{k=1}^{m} \left(\log_k \frac{1}{\rho}\right)^{-t_k},$$

where $0 \leqslant d_1 < n - \alpha d_2 \leqslant n$ and Q_1, Q_2 are constants independent of ρ. Then, for $0 < u \leqslant \max\left[1, \frac{d_2 - (n-\alpha)}{(n-\alpha) - d_2}\right]$, we have

$$v[(g_\alpha * \mu)^u \lambda, \rho] \leqslant Q\left[\rho^{\alpha - n + d_1} \prod_{k=1}^{m} \left(\log_k \frac{1}{\rho}\right)^{-S_k}\right]^u \rho^{d_2} \prod_{a=1}^{b} \left(\log_a \frac{1}{\rho}\right)^{-t_a},\ 0 < \rho < \rho_0,$$

where Q is independent of ρ.

Hence, for $d_1 = n - \alpha p$, $s_k = q_k - 1$, $(k = 1, \ldots, m)$ $d_2 = n$ and $t_a = 0$ $(a = 1, \ldots, b)$, we obtain the

C o r o l l a r y. *Let $\alpha p \leqslant n$ and $\mu \in L_1^+$; suppose that*

$$v(\mu, \rho) \leqslant Q_1 \rho^{n - \alpha p} \prod_{k=1}^{m} \left(\log_k \frac{1}{\rho}\right)^{1 - q_k},\qquad 0 < \rho \leqslant \rho_0 < 1,$$

where Q_1 is a constant independent of ρ.

Then

$$v[g_\alpha * \mu)^{\frac{1}{p-1}}, \rho] \leqslant Q\, \rho^{n-\alpha} \prod_{k=1}^{m} \left(\log_k \frac{1}{\rho}\right)^{\frac{q_k - 1}{p-1}},\qquad 0 < \rho \leqslant \rho_0,$$

where Q is a constant independent of ρ and ρ_0 is sufficiently small.

Remark. This corollary is an extension of lemma 4.2 in D. ADAMS and N. MEYERS' paper [2].

L e m m a 6. *Let $\alpha < n$ and $\lambda \in L_1^+$; suppose*

$$v(\lambda, \rho) \leqslant Q_1 \rho^{n-\alpha} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{\rho}\right)^{-1} \left(\log_m \frac{1}{\rho}\right)^{-s},\qquad 0 < \rho \leqslant \rho_0 < 1,$$

where $s > 1$, Q_1 is a constant independent of ρ and ρ_0 is sufficiently small. Then $g_\alpha * \lambda$ is a bounded function.

Define

$$\tilde{g}_\alpha(r) = \begin{cases} g_\alpha(r) & \text{for } 0 \leqslant r < \rho_0, \\ 0 & \text{for } \qquad r > \rho_0. \end{cases}$$

Rewrite

$$g_\alpha * \lambda(x) = \tilde{g}_\alpha * \lambda(x) + (g_\alpha - \tilde{g}_\alpha) * \lambda(x). \tag{14}$$

But

$$g_\alpha * \lambda(x) = \int \tilde{g}_\alpha(x-y)d\lambda(y) = \int_{B(x,\rho_0)} g_\alpha(x-y)d\lambda(y) \leqslant \text{const.} \int_0^{\rho_0} r^{\alpha-n} \cdot dv(\lambda, x, r)$$

$$\leqslant \text{const.} \int_0^{\rho_0} r^{\alpha-n} d\left[r^{n-\alpha} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{-1} \left(\log_m \frac{1}{r}\right)^{-s}\right] =$$

$$= \text{const.} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{\rho_0}\right)^{-1} \left(\log_m \frac{1}{\rho_0}\right)^{-s} - \text{const.} \int_0^{\rho_0} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{-1} \left(\log_m \frac{1}{r}\right)^{-s} \frac{dr}{r} =$$

$$= \text{const.} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{\rho_0}\right)^{-1} \left(\log_m \frac{1}{\rho_0}\right)^{-s} - \text{const.} \left(\log_m \frac{1}{\rho_0}\right)^{1-s} <$$

$$< \text{const.} \cdot \prod_{k=1}^{m-1} \left(\log_k \frac{1}{\rho_0}\right)^{-1} \left(\log_m \frac{1}{\rho_0}\right)^{-s} < \infty.$$

The second term of the right part of (14) is the convolution of a bounded function with a measure of finite total variation and is thus also bounded, as desired.

Remark. This is an extension of lemma **4.3.** of D. ADAMS and N. MEYERS [2].

THEOREM 4. *Let* $\alpha p \leqslant n$. *Then*

$$H_h \prec H_{h_{\alpha p,\, p-1,\, m,\, \beta}} \prec B_{\alpha, p}\,,$$

where $h(r) = r^{n-\alpha p}\left(\log \frac{1}{r}\right)^{1-q}$ *for* $q > p$ *and* $r < \frac{1}{e}$, *and where*

$$h_{\alpha p,\, p-1,\, m,\, \beta}(r) = r^{n-\alpha p} \prod_{k=1}^{m-1}\left(\log_k \frac{1}{r}\right)^{1-p}\left(\log_m \frac{1}{r}\right)^{-\beta} \qquad (m = 1, 2, \ldots)$$

$\forall \beta > p - 1$ *and* $0 < r < r_m$ *with* $\log_m \frac{1}{r_m} > 1$.

First, let E' be an analytic set with $H_{h_{\alpha p,\, p-1,\, m,\, \beta}}(E') > 0$; then, from the preceding proposition, there exists $\mu \neq 0$ such that $\mu \in \mathfrak{M}^+(E)$ and

$$v(\mu, r) \leqslant r^{n-\alpha p} \prod_{k=1}^{m-1}\left(\log_k \frac{1}{r}\right)^{1-p}\left(\log_m \frac{1}{r}\right)^{-\beta} \qquad (m = 1, 2, \ldots).$$

We may assume that μ has a compact support. Then, by the preceding corollary, where $q_k = p$ $(k = 1, \ldots, m-1)$ and $q_m > p$, it follows that, taking $\lambda = (g_\alpha * \mu)^{\frac{1}{p-1}}$, we are in the hypotheses of the preceding lemma, which allows us to conclude that $g_\alpha * (g_\alpha * \mu)^{\frac{1}{p-1}}$ is bounded in R^n by a constant $M < \infty$. But since $\frac{\mu}{M^{p-1}} \neq 0$ and $g_\alpha\left(g_\alpha * \frac{\mu}{M^{p-1}}\right)^{\frac{1}{p-1}}(x) \leqslant 1$, it follows that $\tilde{b}_{\alpha,p}(E') > 0$ and, on account of proposition 7, $B_{\alpha,p}(E') > 0$. To prove the implication for general E, note that from the definition of Hausdorff measure there exists $E'' \subset E$ such that $H_h(E'') = H_h(E)$ and E'' is a G_δ-set (i.e. a set which is the intersection of a countable family of open sets). Since $B_{\alpha,p}$ is an outer capacity, we may simultaneously choose E'' so that $B_{\alpha,p}(E'') = B_{\alpha,p}(E)$, hence $H_{h_{\alpha p,\, p-1,\, m,\, \beta}}(E) > 0$ yields $B_{\alpha,p}(E) > 0$, hence $H_{h_{\alpha p,\, p=1,\, m,\, \beta}} \prec B_{\alpha,p}$, as desired.

The inclusion $H_h \prec H_{h_{\alpha p,\, p-1,\, m,\, \beta}}$ is a direct consequence of the relation

$$\lim_{r\to 0} \frac{h(r)}{h_{\alpha p,\, p-1,\, m,\, \beta}(r)} = \lim_{r \to 0} \frac{\left(\log \frac{1}{r}\right)^{1-q}}{\prod_{k=1}^{m-1}\left(\log_k \frac{1}{r}\right)^{1-p}\left(\log_m \frac{1}{r}\right)^{-\beta}} = 0 \text{ for } q > p.$$

Remark. The assertion of this theorem for $p \geqslant 2$ is a direct consequence of proposition 6 and theorem 3, however its contribution is

new in the case $1 < p < 2$. This theorem generalizes theorem **4.3** of D. ADAMS and N. MEYERS [2].

Corollary. Cap $E = 0 \Rightarrow H_{h_{n-1,m,\beta}}(E) = 0$ $\forall \beta > n - 1$ $(m = 1, 2, \ldots)$, *where* $h_{n-1,m,\beta}(r) = \prod_{k=1}^{m-1}\left(\log_k \frac{1}{r}\right)^{1-n}\left(\log_m \frac{1}{r}\right)^{-\beta}$.

This is a consequence of the preceding theorem and of the corollary of proposition 4. But it may be obtained also from

Proposition 9. Cap $E = 0 \Rightarrow H_h(E) = 0$ *for all measure functions* h *satisfying*

$$\int_0^{r_0} h(r)^{\frac{1}{n-1}} \frac{dr}{r} < \infty$$

for $0 < r_0 < 1$ *sufficiently small.*

For the proof, see H. WALLIN ([23], theorem 4.1) or D. ADAMS ([1], theorem 2).

In order to show that the preceding theorem is best possible, we recall some definitions and previous results.

Proposition 10. *Let* h *be a measure function satisfying*

(15) $$h(2r) \leqslant Q h(r),$$

let us say for $0 \leqslant r \leqslant \frac{1}{2}$ *and where* $0 < Q < \infty$. *If* F *is a Cantor set in* R^n, *then*

$$\frac{1}{Q} \varliminf_{m \to \infty} 2^{nm} h(l_m) \leqslant H_h(F) \leqslant Q \varliminf_{m\to\infty} 2^{nm} h(l_m).$$

(For the proof, see D. ADAMS and N. MEYERS [2], proposition 5.1.)

A set $E \subset R^n$ is said to have *a lower spherical h-density at a point* x if

(16) $$\varliminf_{r \to 0} \frac{H_h[B(x,r) \cap E]}{h(2r)} > 0.$$

Proposition 11. *The set* E *has lower spherical h-density at* $\forall x \in E$, *where* E *is a Borel set satisfying* $0 < H_h(E) < \infty$, *if, for* $0 < r < r_1$ *(*r_1 *sufficiently small),* (15) *holds, where the constant* Q *satisfies* $1 \leqslant Q < 2^n$ *and* E *is a set of Cantor type* $E = \bigcap_{m=1}^{\infty} E_m$, *with* E_m *obtained in the* m^{th} *step, consisting of* 2^{mn} *n-dimensional intervals with edges of length* l_m, $2l_{m+1} < l_m$, *obtained in the usual way satisfying the following double inequality*

$$c_1 < 2^{nm} h(l_m) < c_2, \quad c_1, c_2 \text{ constants.}$$

(For the proof, see H. WALLIN [23] as remark 4.1, or our paper [7], lemma 16.)

P r o p o s i t i o n 12. *There are constants $a, b > 0$, independent of x and μ such that*

$$g_\alpha * (g_\alpha * \mu)^{\frac{1}{p-1}}(x) \geqslant \frac{1}{a}\int_0^\infty [r^{\alpha p-n} v(\mu, x, r)]^{\frac{1}{p-1}} e^{\frac{r}{b}} \frac{dr}{r}.$$

(For the proof, see D. ADAMS [1], theorem 2.)

L e m m a 7. *Let h be a measure function such that*

$$\int_0^R [r^{\alpha p-n} h(r)]^{\frac{1}{p-1}} \frac{dr}{r} = \infty$$

and let E' be a Borel set satisfying $0 < H_h(E') < \infty$ such that E' has positive lower spherical h-density at $\forall x \in E'$. Then $B_{\alpha,p}(E') = 0$.

Let a measure μ be defined as

$$\mu(E) = H_h(E \cap E') \ \forall \text{ Borel set } E.$$

According to (16), there are, for any fixed $x \in E$, numbers $c > 0$ and $r_1 > 0$ such that

$$v(\mu, x, r) = H_h[B(x, r) \cap E] > ch(2r) \text{ for } 0 < r < r_1.$$

Consequently, on account of the preceding proposition,

$$g_\alpha * (g_\alpha * \mu)^{\frac{1}{p-1}}(x) \geqq \frac{1}{a}\int_0^{r_1} [r^{\alpha p-n} v(\mu, x, r)]^{\frac{1}{p-1}} e^{-\frac{r}{b}} \frac{dr}{r} >$$

$$> \frac{c^{\frac{1}{p-1}}}{a} e^{-\frac{r_1}{b}} \int_0^{r_1} [r^{\alpha p-n} h(2r)]^{\frac{1}{p-1}} \frac{dr}{r} = \infty,$$

as desired.

THEOREM 5. *There exist compact sets $F \subset R^n$ with $B_{\alpha,p}(F) = 0$, $\alpha p \leqq n$, but with $0 < H_{h_{\alpha p, p-1, m, \beta}}(F) < \infty \ \forall \beta \leqq p-1$ $(m = 1, 2, \ldots)$.*

Let F be an n-dimensional set of Cantor type, $F = \bigcap_{q=1}^{\infty} F_q$, where F_q, obtained in the q^{th} step, consists of 2^{qn} n-dimensional intervals with edges of length l_q, $2l_{q+1} < l_q$, obtained in the usual way and l_q is chosen such that

$$(17) \qquad c_1 < 2^{qn} h_{\alpha p, p-1, m, \beta}(l_q) < c_2 \quad (c_1, c_2 \text{ constants}).$$

We shall establish that $h_{\alpha p, p-1, m, \beta}$ satisfies (15), i.e. that

$$(18) \quad (2r)^{\alpha p-n} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{2r}\right)^{1-p} \left(\log_m \frac{1}{2r}\right)^{-\beta} \leqq Q r^{\alpha p-n} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{1-p} \left(\log_m \frac{1}{r}\right)^{-\beta},$$

with $0 < Q < 2^n$. Let us take Q of the form $Q = Q_m^\beta \prod_{k=1}^{m-1} Q_k^{p-1}$. It is enough to prove that

$$\log_k \frac{1}{r} \leqq Q_k \log \frac{1}{2r} \qquad (k = 1, \ldots, m).$$

We use an induction argument. It is easy to verify that this inequality holds for $k = 1$ with $Q_1 = 1 + \varepsilon$ ($\varepsilon > 0$ as small as one pleases), and suppose it is true for $k - 1$. Then

$$\log_k \frac{1}{r} = \log\left(\log_{k-1} \frac{1}{r}\right) \leqq \log\left(Q_{k-1} \log_{k-1} \frac{1}{2r}\right) \leqq \log\left[\left(\log_{k-1} \frac{1}{2r}\right)^{Q_k}\right] =$$

$$= Q_k \log_k \frac{1}{2r}$$

since, for r sufficiently small and $Q_k > 1$,

$$1 \leqq Q_{k-1} \leqq \left(\log_{k-1} \frac{1}{2r}\right)^{Q_{k-1}}.$$

For $\varepsilon > 0$ sufficiently small, $1 < Q < 2^n$ and we ar in the hypotheses of proposition 11 asserting that F is of lower spherical h-density. Next, according to (17) and taking into account proposition 10, it follows that $0 < H_{\alpha p, p-1, m, \beta}(F) < \infty$ and since, for r_0 small enough,

$$\int_0^{r_0} [r^{\alpha p-n} h_{\alpha p, p-1, m, \beta}(r)]^{\frac{1}{p-1}} \frac{dr}{r} = \int_0^{r_0} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{-1} \left(\log_m \frac{1}{r}\right)^{-\frac{\beta}{p-1}} \frac{dr}{r} = \infty \ \forall \beta \leqslant p-1$$

and F is of positive lower spherical h-density, the preceding lemma implies $B_{\alpha,p}(F) = 0$, as desired, allowing us to conclude that the result given by theorem 4 is best possible.

C o r o l l a r y. *There exist compact sets $F \subset R^n$ of conformal capacity zero, but with $0 < H_{h_{n,n-1,m,\beta}}(F) < \infty \ \forall \beta \leqq n-1$, where $h_{n-1,m,\beta}$ is defined in the preceding corollary.*

This corollary shows that, in particular, also the preceding corollary is best possible.

And now, let us show that also the result expressed by the inequality (13) of theorem 3 and its corollary are best possible in a certain sense.

THEOREM 6. *There exist compact sets $F \subset R^n$ with $B_{\alpha,p}(F) = 0$, $\alpha p \leqq n$, but with $B_{(\alpha p, p-2, m, \beta)}(F) > 0 \ \forall \beta < p-2$ $(m = 1, 2, \ldots)$.*

According to the preceding theorem, there exist Cantor sets $F \cup R^n$, with $B_{\alpha,p}(F) = 0$ and $0 < H_{h_{\alpha p, p-1, m, \beta}}(F) < \infty$ $\forall \beta \leqslant p - 1$ $(m = 1, 2, \ldots)$.

On the other hand, from proposition 6, we deduce that if a set E has the Hausdorff h-measure $H_h(E) > 0$ and $\int_0^{r_0} \Phi(r) dh(r) < \infty$ for a sufficiently smal r_0 then c^*_Φ $(E) > 0$. But if $h = h_{\alpha p, p-1, m, \beta}$ and $\Phi = g_{\alpha p, p-2, m, \gamma}$, we get

$$\int_0^{r_0} r^{\alpha p - n} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{p-2} \left(\log_m \frac{1}{r}\right)^{\gamma} d\left[r^{n-\alpha p} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{1-p} \left(\log_m \frac{1}{r}\right)^{-\beta}\right] =$$

$$= (n - \alpha p) \int_0^{r_0} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{-1} \left(\log_m \frac{1}{r}\right)^{\gamma-\beta} \frac{dr}{r} +$$

$$+ (p-1) \int_0^{r_0} \left(\log \frac{1}{r}\right)^{-2} \prod_{k=2}^{m-1} \left(\log_k \frac{1}{r}\right)^{-1} \left(\log_m \frac{1}{r}\right)^{\gamma-\beta} \frac{dr}{r} + \ldots +$$

$$+ (p-1) \int_0^{r_0} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{-2} \left(\log_m \frac{1}{r}\right)^{\gamma-\beta} \frac{dr}{r} + \beta \int_0^{r_0} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{-2} \cdot$$

$$\cdot \left(\log_m \frac{1}{r}\right)^{\gamma-\beta+1} \frac{dr}{r} \leqslant \frac{(n-\alpha p)}{\gamma-\beta+1} u^{\gamma-\beta+1}\Big|_{\log_m \frac{1}{r_0}}^{\infty} + \frac{(m-1)(p-1)}{\gamma-\beta+1} u^{\gamma-\beta+1}\Big|_{\log_m \frac{1}{r_0}}^{\infty} +$$

$$+ \frac{\beta}{\gamma-\beta} u^{\gamma-\beta}\Big|_{\log_m \frac{1}{r_0}}^{\infty} < \infty$$

for $\gamma < \beta - 1 \leqslant p - 2$, and in the case $\alpha p = n$, for $\Phi = g_{n, p-2, m, \gamma}$, we have

$$\int_0^{r_0} \Phi(r)\, dh(r) = \int_0^{r_0} \left(\log \frac{1}{r}\right)^{p-1} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{p-2} \left(\log_m \frac{1}{r}\right)^{\gamma} d\left[\prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{1-p} \cdot\right.$$

$$\left.\cdot \left(\log_m \frac{1}{r}\right)^{-\beta}\right] = (p-1) \int_{\log \frac{1}{r_0}}^{\infty} \prod_{k=1}^{m-2} (\log_k u)^{-1} (\log_{m-1} u)^{\gamma-\beta} \frac{du}{u} +$$

$$+ (p-1) \int_{\log_2 \frac{1}{r_0}}^{\infty} u^{-2} \prod_{k=1}^{m-3} (\log_k u)^{-1} (\log_{m-2} u)^{\gamma-\beta} du + \ldots +$$

$$+ (p-1) \int_{\log_2 \frac{1}{r_0}}^{\infty} \prod_{k=1}^{m-3} (\log_k u)^{-2} (\log_{m-2} u)^{\gamma-\beta-1} du \leqslant \frac{p-1}{\gamma-\beta+1} v^{\gamma-\beta+1}\Big|_{\log_m \frac{1}{r_0}}^{\infty} +$$

$$+ \frac{p-1}{\gamma-\beta+1} v^{\gamma-\beta+1}\Big|_{\log_m \frac{1}{r_m}}^{\infty} + \ldots + \frac{\beta}{\gamma-\beta} v^{\gamma-\beta}\Big|_{\log_m \frac{1}{r_m}}^{\infty} <$$

$$< \infty, \ \forall \gamma < \beta - 1 \leqslant p - 2,$$

as desired.

Corollary. *There exist compact sets $F \subset R^n$ with* cap $F = 0$, *but with $B_{(n, n-2, m, \beta)}(F) > 0$ $\forall \beta < n - 2$ $(m = 1, 2, \ldots)$.*

Now, we shall give an extension for $p = 1$ of some results obtained by N. MEYERS [12] for $p > 1$. Let us denote $c_{g_\alpha} = b_\alpha = b_{\alpha,1}$ and $c^*_{g_\alpha} = B_\alpha = B_{\alpha,1}$.

We give first some preliminary results.

Proposition 13. *Let $\alpha p < n$, $p \geqslant 1$, $\frac{1}{q} = \frac{n-\alpha}{n} - \frac{1}{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$ and*

$$u^R_\alpha(x) = \int_{r \leqslant R} |x - y|^{\alpha - n} f(y)\, dy.$$

Then

$$||u^R_\alpha||_{q^*} \leqslant K||f||_p,$$

where $K = K(R, p, q, q^*, n)$ and $p < q^* < q$.

(For the proof, see S. SOBOLEV [18], p. 481.)

Lemma 8. *Let $\frac{1}{q} = \frac{n-\alpha}{n}$ and $\alpha < n$, then*

$$||g_\alpha * f||_{q^*} \leqslant K||f||_1,$$

where $1 < q^* < q$ and $K = K(\alpha, q^*, n, R)$ with $0 < R < \infty$.

Let us denote

$$\tilde{g}_\alpha(r) = \begin{cases} g_\alpha(r) & \text{for } r \leqslant R, \\ 0 & \text{for } r > R. \end{cases}$$

Next,

$$g_\alpha * f(x) = \tilde{g}_\alpha * f(x) + (g_\alpha - \tilde{g}_\alpha) * f(x),$$

hence, according to the preceding proposition,

(19) $||g_\alpha * f||_{q^*} \leqslant ||\tilde{g}_\alpha * f||_{q^*} + ||(g_\alpha - \tilde{g}_\alpha) * f||_{q^*} \leqslant K_1||f||_1 + ||(g_\alpha - \tilde{g}_\alpha) * f||_{q^*}.$

But, applying MINKOWSKI 's inequality (see G. H. HARDY, J. E. LITTLEWOOD and G. POLYA [11], theorem 202, p. 148):

$$\{\int[\int h(x, y)dy]^{q^*} dx\}^{\frac{1}{q^*}} \leqslant \int[\int h(x, y)^{q^*} dx]^{\frac{1}{q^*}}dy \qquad (q^* > 1),$$

we obtain

$$(20) \quad \|(g_\alpha - \widetilde{g}_\alpha) * f\|_{q^*} = (\int[\int [g_\alpha(x-y) - \widetilde{g}_\alpha(x-y)]\, f(y)\, dy\}^{q^*} dx)^{\frac{1}{q^*}} \leqslant$$

$$\leqslant \int\{\int[g_\alpha(x-y) - \widetilde{g}_\alpha(x-y)]^{q^*} dx\}^{\frac{1}{q^*}} f(y)\, dy = \int\Big[\int_{|x-y|>R} g_\alpha(x-y)^{q^*} dx\Big]^{\frac{1}{q^*}} \cdot$$

$$\cdot f(y)\, dy = \int\Big[\int_{|z|>R} g_\alpha(z)^{q^*} dz\Big]^{\frac{1}{q^*}} f(y)\, dy \leqslant \Big[\text{const.}\int_R^\infty r^{\frac{\alpha-n-1}{2}q^*+n-1} e^{-q^* r}\, dr\Big]^{\frac{1}{q^*}} \cdot \|f\|_1$$

$$= K_2 \|f\|_1 < \infty$$

Combining (19) with (20), it follows that

$$\|g_\alpha * f\|_{q^*} \leqslant [K_1 + K_2]\, \|f\|_1 = K\, \|f\|_1,$$

as desired.

Proposition 14. *If* $\alpha p < n$, $p > 1$, $\frac{1}{q} = \frac{n-\alpha}{n} - \frac{1}{p'}$ *and*

$$u_\alpha(x) = \int |x - y|^{\alpha-n} f(y)\, dy,$$

then

$$\|u_\alpha\|_q \leqslant K\, \|f\|_p,$$

where $K = K(\alpha, p, q)$.

(For the proof, see S. SOBOLEV [18], p. 481.)

Argiung as in theorem 20 of N. MEYERS' paper [12], and using (in the proof) the preceding lemma instead of the preceding proposition, we get

Lemma 9. *The following relations hold:*

(i) *If* $\alpha < n$, *then* $B_\alpha(E) \geqslant \chi(m^* E)^{q^*}$, $1 < q^* < \frac{n}{n-\alpha}$;

(ii) $B_n(E) \geqslant \chi(m^* E)^\varepsilon$ *for* $0 < \varepsilon \leqslant 1$.

In each case, χ *is a constant independent of the set* E, *but depending of the numerical parameters present.*

Arguing as N. Meyers [12] in his theorem 21, we obtain

Lemma 10. *If* $\alpha < n$, *then there exists a finite constant* $\chi > 0$, *independent of* ρ *such that*

$$\chi\rho^{n-\alpha} \leqslant B_\alpha[B(x_0, \rho)] \leqslant \chi\rho^{n-\alpha} \quad \text{for } 0 < \rho \leqslant 1.$$

Lemma 11. *There exists a constant* $0 < \chi < \infty$, *independent of* ρ, *such that*

$$(21) \qquad \chi^{-1}\, \Phi(\rho)^{-1} \leqslant c_\Phi[\overline{B(x_0, \rho)}] \leqslant \Phi(2\rho)^{-1},$$

where, for $m = 1, 2, \ldots$ *and* $\forall\beta > 0$,

$$\Phi(r) = \Phi_{\alpha p, p-2, m, \beta}(r) = \begin{cases} r^{\alpha p - n}\prod_{k=1}^{m-1}\left(\log_k \frac{1}{r}\right)^{p-2}\left(\log_m \frac{1}{r}\right)^\beta \text{ for } r \leqq r_m, \\ r^{\alpha p - n}\prod_{k=1}^{m-1}\left(\log_k \frac{1}{r}\right)^{p-2}\left(\log_m \frac{1}{r}\right)^\beta \text{ for } r > r_m, \end{cases}$$

where $\alpha p < n$, *or*

$$\Phi(r) = \Phi_{n, p-2, m, \beta}(r) = \begin{cases} \left(\log \frac{1}{r}\right)^{p-1}\prod_{k=2}^{m-1}\left(\log_k \frac{1}{r}\right)^{p-2}\left(\log_m \frac{1}{r}\right)^\beta \text{ for } r \leqq r_m, \\ \left(\log \frac{1}{r_m}\right)^{p-1}\prod_{k=1}^{m-1}\left(\log_k \frac{1}{r_m}\right)^{p-2}\left(\log_m \frac{1}{r_m}\right)^\beta \text{ for } r > r_m. \end{cases}$$

Since the capacity of a ball of radius $\rho > 0$ is independent of the position of its centre, we may assume for simplicity's sake $x_0 = 0$. Then, let ν_ρ be the c_Φ-capacitary distribution of $B(\rho) = B(0, \rho)$ for the kernel Φ from above. Then, $\sup_x \Phi * \nu_\rho(x) = 1$, so that, since $S_{\nu_\rho} \subset B(\rho)$ and for $|x| = \rho$,

$$1 \geqq \int_{B(\rho)} \Phi(x - y)d\nu_\rho(y) \geqq \int_{B(\rho)} \Phi(2|x|)d\nu_\rho(y) = \Phi(2\rho)\, \nu_\rho[B(\rho)] = \Phi(2\rho)c_\Phi[B(\rho)],$$

yielding the second part of (21).

We now derive the lower bound of $c_\Phi[\overline{B(\rho)}]$, which, according to propositon 2, is equal to $C_{\Phi,1}[\overline{B(\rho)}]$. To this end, let $m_\rho = m_{|B(\rho)}$, where m is the m-dimensional Lebesgue measure. Then,

$$(22) \qquad \Phi * m_\rho(x) = \int_{B(\rho)} \Phi(x - y)dy \leqq \int_{B(\rho)} \Phi(y)dy = \text{const.}\int_0^\rho \Phi(|y|\, |y|^{n-1}d|y| \leqq$$

$$\leqq \text{const.}\ \Phi(\rho)\rho^n.$$

Next, for a test function f for the kernel $C_{\Phi,1}[\overline{B(\rho)}]$, we have

$$\text{const. } \rho^n = \int dm_\rho(y) \leqq \int_{B(\rho)} \Phi * f(y)dy \leqq \int\int_{B(\rho)} \Phi(x-y)f(x)dxdy =$$

$$= \int f(x)[\int \Phi(x-y)dm_\rho(y)]dx \leqq \|f\|_1 \sup_x \Phi * m_\rho(x),$$

hence and on account of (22), it follows that

$$\frac{\text{const. } \rho^n}{\|f\|_1} \leqq \sup_x \Phi * m_\rho(x) \leqq \text{const. } \Phi(\rho)\rho^n,$$

whence

$$\|f\|_1 \geqq \text{const. } \Phi(\rho)^{-1}$$

and, since f was an arbitrary test function, we may take the infimum for such functions, obtaining in this way also the first part of (21).

Lemma 12. *Let $E \subset R^n$ be a bounded set and $\alpha p \leqq n$, then*

$$\text{(23)} \qquad \text{const. } B_{(\alpha p,\, p-2,\, m,\, \beta)}(E) \leqq C^*_{\Phi_{\alpha p,\, p-2,\, m,\, \beta}}(E) \leqq \text{const. } B_{(\alpha p,\, p-2,\, m,\, \beta)}(E)$$

$\forall \beta > 0$ and $m = 1, 2, \ldots$.

It is clear that

$$g_{\alpha p,\, p-2,\, m,\, \beta}(r) \leqq Q\Phi_{\alpha p,\, p-2,\, m,\, \beta}(r),$$

where Q is a constant independent of r. Therefore, $\forall E \subset R^n$, the second part of (23) holds.

We now seek the opposite inequality, which we shall establish for $C_{\Phi,1}$ since, by proposition 3, $C^*_\Phi = C_{\Phi,1}$. Evident, the diameter $d = d(E) < \infty$, because E is bounded. Next, the translational invariance of the capacities allows us to suppose (without loss of generality) that $E \subset B(d)$. Let f be a test function for $C_{\Phi,1}(E)$. Since, according to the preceding lemma and proposition 3,

$$C_{\Phi,1}[\overline{B(\rho)}] \leqq \Phi(2d)^{-1},$$

we may assume that $\|f\|_1 \leqq 2\Phi(2d)^{-1}$. Hence, if $a = \left(\frac{1}{4}\right)^{\frac{1}{p-2}}$ and $\Phi = \Phi_{\alpha p,\, p-2,\, m,\, \beta}$ $(\alpha p < n)$, then,

$$\int_{|y| \geqq d + (2d)a} \Phi(x-y)f(y)dy \leqq \Phi[(2d)^a]\|f\|_1 \leqq 2\Phi[(2d)a]\Phi(2d)^{-1} =$$

$$= \frac{1}{2}(2d)^{a(\alpha p - n)}\left(\log\frac{1}{2d}\right)^{p-2}\prod_{k=2}^{m-1}\left\{\log_k\left[\left(\frac{1}{2d}\right)a\right]\right\}^{p-2}\left\{\log_m\left[\left(\frac{1}{2d}\right)a\right]\right\}^{\beta}(2d)^{n-\alpha p} \times$$

$$\times \prod_{k=1}^{m-1}\left(\log_k\frac{1}{2d}\right)^{2-p}\left(\log_m\frac{1}{2d}\right)^{-\beta} < \frac{1}{2} \qquad \forall x \in E.$$

Therefore

$$\int_{|y| < d + (2d)a} 2\Phi(x-y)f(y)dy \geqq 1 \qquad \forall x \in E.$$

Since

$$\Phi(r) = \Phi_{\alpha p,\, p-2,\, m,\, \beta}(r) \leqq Qg_{\alpha p,\, p-2,\, m,\, \beta}(r) \text{ for } r < 2d + (2d)a,$$

where Q is constant, we conclude that $2Qf$ is a test function for $B_{(\alpha p,\, p-2,\, m,\, \beta)}(E)$; thus

$$B_{(\alpha p,\, p-2,\, m,\, \beta)}(E) \leqq \text{const. } C_{\Phi_{\alpha p,\, p-2,\, m,\, \beta}}(E).$$

The same argument still holds in the case $\Phi = \Phi_{n,\, p-2,\, m,\, \beta}$; we have only to take this time $a = \left(\frac{1}{4}\right)^{p-1}$.

Now, let us remind *Ugaheri's maximum principle* (UGAHERI [22]). There exists a constant $\lambda > 0$ such that

$$\sup_{x \in R^n} u^\mu_\Phi(x) \leqq \lambda \sup_{y \in S_\mu} u^\mu_\Phi(y) \qquad \forall \mu \in \mathfrak{M}^+,$$

where u^μ_Φ is the potential of kernel Φ with respect to the measure μ:

$$u^\mu_\Phi(x) = \int \Phi(x-y)\, d\mu(y).$$

G. CHOQUET [9] calls it ,,*principe du maximum λ-dilaté*".

Proposition 15. *The kernel $\Phi(x-y)$ satisfies Ugaheri's maximum principle if there exists $\lambda > 0$ such that $|x_2 - y| \leqq 2|x_1 - y| \Rightarrow \Phi(x_1 - y) \leqq \lambda\Phi(x_2 - y)$.*

(For the proof, see G. CHOQUET [9], critere 3, p. 637.)

Corollary. *The kernel $\Phi_{\alpha p,\, p-2,\, m,\, \beta}$ $(\alpha p \leqq n,\ m = 1, 2, \ldots)$ verifies Ugaheri's maximum principle.*

Since

$$|x_1 - y|^{\alpha p - n}\prod_{k=1}^{m-1}\left(\log_k\frac{1}{|x_1 - y|}\right)^{p-2}\left(\log_m\frac{1}{|x_1 - y|}\right)^{\beta} \leqq$$

$$\leqq \left(\frac{|x_2 - y|}{2}\right)^{\alpha p - n}\prod_{k=1}^{m-1}\left(\log_k\frac{2}{|x_2 - y|}\right)^{p-2}\left(\log_m\frac{2}{|x_2 - y|}\right)^{\beta};$$

proposition 15 holds as a consequence of inequality (18).

Proposition 16. *Given a compact set $F \subset R^n$, there exists $\nu \in \mathfrak{M}^+$, with $S_\nu \subset F$ and $\nu(F) = 1$ such that $u^\nu_\Phi(x) \geqq C^-_\Phi(F)$ p.p. on F (i.e. except on a subset of F with the Φ-capacity zero), $u^\nu_\Phi(x) \leqq C^-_\Phi(F)$ on S_ν and $u^\nu_\Phi(x) \leqq \lambda C^-_\Phi(F)$ everywhere.*

(For the proof, see L. CARLESON 's theorem 3, p. 17 in [8].)

Remark. In Carleson's theorem, one finds the stronger conclusion $u^{\check{\nu}}_{\Phi}(x) \leqq C_{\Phi}^{-1}(F)$ everywhere, but this a consequence of more restrictive conditions on the kernel. In our case, instead of Frostman's maximum principle, only Ugaheri's holds.

C o r o l l a r y. *For any kernel* Φ *and any compact set* $F \subset R^n$, *there exists a measure* $\mu \in \mathfrak{M}^+$ *with* $S_\mu \subset F$ *so that*

(a) $u^{\mu}_{\Phi}(x) = 1$ *p.p. on* S_μ,

(b) $u^{\mu}_{\Phi}(x) \geqq 1$ *p.p. on* F,

(c) $u^{\mu}_{\Phi}(x) \leqq \lambda$ *everywhere and*

(d) $\mu(F) = c_{\Phi}(F)$.

As a consequence of Fubini theorem, we have for symmetric kernels [i.e. such that $\Phi(x, y) = \Phi(y, x)$]

P r o p o s i t i o n 17. $\int u^{\mu}_{\Phi}(x)\, d\nu(x) = \int u^{\check{\nu}}_{\Phi}(x)\, d\mu(x)$.

L e m m a 13. *For any compact set* $F \subset R^n$,

$$\frac{1}{\lambda} C_{\Phi}(F) \leqq c_{\Phi}(F) \leqq \lambda C_{\Phi}(F).$$

Let μ be the measure of the preceding corollary and ν of proposition 16. Then, on account of the preceding corollary and proposition 16 and 17 we deduce that

$$\frac{c_{\Phi}(F)}{C_{\Phi}(F)} = \int \frac{d\mu(x)}{C_{\Phi}(F)} = \int_F \frac{d\mu(x)}{C_{\Phi}(F)} \leqq \int_F u^{\check{\nu}}_{\Phi}(x)\, d\mu(x) = \int u^{\check{\nu}}_{\Phi}(x)\, d\mu(x) = \int u^{\mu}_{\Phi}(x)\, d\nu(x) \leqq$$

$$\leqq \lambda\nu(F) = \lambda,$$

hence $c_{\Phi}(F) \leqq \lambda C_{\Phi}(F)$.

For the opposite inequality, using again the preceding corollary and propositions 16 and 17 we get

$$\frac{c_{\Phi}(F)}{C_{\Phi}(F)} \geqq \frac{1}{\lambda} \int u^{\check{\nu}}_{\Phi}(x)\, d\mu(x) = \frac{1}{\lambda} \int u^{\mu}_{\Phi}(x)\, d\nu(x) \geqq \frac{1}{\lambda},$$

hence, $C_{\Phi}(F) \leqq \lambda c_{\Phi}(F)$, as desired.

And now, as a consequence of the preceding 2 lemmas, all the results of this paper with respect to the capacity c_{Φ} and, in particular, with respect to the Bessel capacities of the form $B_{(\alpha p,\, p-2,\, m,\, \beta)}$ still hold for the Φ-capacity C_{Φ}. In particular we deduce

C o r o l l a r y 1. *Let* $\alpha p \leqq n$, $p \geqq 2$ *and* F *compact, then*

$$C_{\Phi_{\alpha p,\, p-2,\, m,\, \beta}}(F) \leqq Q B_{\alpha, p}(F) \quad \forall \beta > p-2 \quad (m = 1, 2, \ldots)$$

but even $\forall \beta \geqq 0$ *in the particular case* $p = 2$, *and there are Cantor sets* F' *such that* $B_{\alpha, p}(F') = 0$, *while*

$$C_{\Phi_{\alpha p,\, p-2,\, m,\, \gamma}}(F') > 0 \quad \forall \gamma < p-2 \quad (m = 1, 2, \ldots).$$

C o r o l l a r y 2. *If* F *is compact and* cap $F = 0$, *then*

$$C_{\Phi_{n,\, n-2,\, m,\, \beta}}(F) = 0 \quad \forall \beta > n-2 \quad (m = 1, 2, \ldots)$$

but, if $n = 2$, F *is of logarithmic, capacity* $C_0(F) = 0$, *where* $C_0 = C_{\Phi}$, $\Phi(r) = \log\frac{1}{r}$ *and there exist Cantor sets* F' *such that* cap $F' = 0$, *while*

$$C_{\Phi_{n,\, n-2,\, m,\, \gamma}}(F') > 0 \quad \forall \gamma < n-2 \quad (m = 1, 2, \ldots).$$

In TAYLOR 's paper [19] on the relations between Hausdorff h-measures and the Φ-capacities C_{Φ}, he obtains the following result:

P r o p o s i t i o n 18. *If the measure functions* $h_{\alpha,\beta}$, $h^0_{\alpha,\beta}$ *and the kernels* $\Phi_{\mu,\nu}$, $\Phi^0_{\mu,\nu}$ *are given by*

$$h^0_{\alpha,\beta}(r) = r^{\alpha}\left(\log\frac{1}{r}\right)^{\beta},\ h_{\alpha,\beta}(r) = \left(\log\frac{1}{r}\right)^{-\alpha}\left(\log_2\frac{1}{r}\right)^{\beta},\ \alpha > 0,$$

$$\Phi^0_{\mu,\nu}(r) = r^{-\mu}\left(\log\frac{1}{r}\right)^{-\nu},\ \Phi_{\mu,\nu}(r) = \left(\log\frac{1}{r}\right)^{\mu}\left(\log_2\frac{1}{r}\right)^{-\nu},\ \mu > 0,$$

then

(A) *if* $\alpha < \mu$ *and* β, ν *are arbitrary or* $\alpha = \mu$ *and* $\beta \geqq \nu$, *then* $H_{h^0_{\alpha,\beta}}(E) < \infty \Rightarrow C_{\Phi^0_{\mu,\nu}}(E) = 0$ *and* $H_{h_{\alpha,\beta}}(E) < \infty \Rightarrow C_{\Phi_{\mu,\nu}}(E) = 0$;

(B) *if* $\alpha > \mu$ *and* β, ν *are arbitrary or* $\alpha = \mu$ *and* $\beta < \nu - 1$; *then* $H_{h^0_{\alpha,\beta}}(E) > 0 \Rightarrow C_{\Phi^0_{\mu,\nu}}(E) > 0$ *and* $H_{h_{\alpha,\beta}}(E) > 0 \Rightarrow C_{\Phi_{\mu,\nu}}(E) > 0$.

But, from proposition 6, we deduce, in general,

L e m m a 14. *Let*

$$h(r) = \prod_{k=0}^{m}\left(\log\frac{1}{r}\right)^{-\alpha_k},$$

$$\Phi(r) = \prod_{k=0}^{m}\left(\log_k\frac{1}{r}\right)^{\beta_k},$$

where $\log_0\frac{1}{r} = \frac{1}{r}$, $\log_1\frac{1}{r} = \log\frac{1}{r}$, $\alpha_i = \beta_i = 0$ $(i = 0, \ldots, a)$ *and* α_{a+1}, $\beta_{a+1} > 0$ *or* α_0, $\beta_0 > 0$ *(in which case we have to consider* $a = -1$**).** *Then*

(A′) *if* $\alpha_{a+1} < \beta_{a+1}$, *or* $\alpha_j = \beta_j$ $(j = a+1, \ldots, a+s \leqq m)$ *and* $\alpha_{a+s+1} < \beta_{a+s+1}$ *(for* $a + s < m$*), then* $H_h(E) < \infty \Rightarrow C_{\Phi}(E) = 0$.

(B′) *if* $\alpha_{a+1} > \beta_{a+1}$, *or* $\alpha_{a+1} = \beta_{a+1}$ *and* $\alpha_{a+2} > \beta_{a+2} + 1$, *or* $\alpha_{a+1} = \beta_{a+1}$, $\alpha_q = \beta_q + 1$ $(q = a+2, \ldots, a+s < m)$ *and* $\alpha_{a+s+1} > \beta_{a+s+1}$, *then* $H_h(E) > 0 \Rightarrow C_{\Phi}(E) > 0$.

Indeed, in the case (A′), $H_h(E) < \infty$ implies $H_{h_1}(E) < \infty$ with

$$h_1(r) = \prod_{k=0}^{\infty}\left(\log_k\frac{1}{r}\right)^{-\beta_k},$$

since

$$\lim_{r\to\infty} \frac{h_1(r)}{h(r)} < \infty$$

and proposition 6 yields $C_\Phi(E) = 0$ because $\Phi(r) = \frac{1}{h_1(r)}$.

In the case (B′), on account of proposition 6, for $\alpha_{a+1} > \beta_{a+1}$ and and $r_0 > 0$ sufficiently small, we have

$$\int_0^{r_0} \Phi(r)\, dh(r) = -\alpha_{a+1} \int_0^{r_0} \left(\log_{a+1} \frac{1}{r}\right)^{\beta_{a+1}-\alpha_{a+1}-1} \left(\log_{a+2} \frac{1}{r}\right)^{\beta_{a+2}-\alpha_{a+2}} \cdots$$

$$\cdots \left(\log_m \frac{1}{r}\right)^{\beta_m-\alpha_m} d\left(\log_{a+1} \frac{1}{r}\right) - \cdots - \alpha_m \int_0^{r_0} \left(\log_{a+1} \frac{1}{r}\right)^{\beta_{a+1}-\alpha_{a+1}} \cdots$$

$$\cdots \left(\log_m \frac{1}{r}\right)^{\beta_m-\alpha_m-1} d\left(\log_{a+1} \frac{1}{r}\right) = \alpha_{a+1} \int_{\log_{a+1}\frac{1}{r_0}}^{\infty} u^{\beta_{a+1}-\alpha_{a+1}-1} (\log u)^{\beta_{a+2}-\alpha_{a+2}} \cdots$$

$$\cdots (\log_{m-a-1} u)^{\beta_m-\alpha_m} du + \cdots + \alpha_m \int_{\log_{a+1}\frac{1}{r_0}}^{\infty} u^{\beta_{a+1}-\alpha_{a+1}-1} \cdots$$

$$\cdots (\log_{m-a-1} u)^{\beta_m-\alpha_m-1}\, du < \alpha_{a+1} \left(\log_{a+1} \frac{1}{r_0}\right)^{\frac{\beta_{a+1}-\alpha_{a+1}}{2}} \left(\log_{a+1} \frac{1}{r_0}\right)^{\beta_{a+2}-\alpha_{a+2}} \cdots$$

$$\cdots \left(\log_m \frac{1}{r_0}\right)^{\beta_m-\alpha_m} \int_{\log_{a+1}\frac{1}{r_0}}^{\infty} u^{\frac{\beta_{a+1}-\alpha_{a+1}-1}{2}} du + \cdots +$$

$$+ \alpha_m \left(\log_{a+1} \frac{1}{r_0}\right)^{\frac{\beta_{a+1}-\alpha_{a+1}}{2}} \left(\log_{a+1} \frac{1}{r_0}\right)^{\beta_{a+2}-\alpha_{a+2}-1} \cdots$$

$$\cdots \left(\log_m \frac{1}{r_0}\right)^{\beta_m-\alpha_m-1} \int_{\log_{a+1}\frac{1}{r_0}}^{\infty} u^{\frac{\beta_{a+1}-\alpha_{a+1}}{2}-1} du = \frac{2\alpha_{a+1}}{\alpha_{a+1}-\beta_{a+1}} \left(\log_{a+1} \frac{1}{r_0}\right)^{\beta_{a+1}-\alpha_{a+1}} \cdots$$

$$\left(\log_m \frac{1}{r_0}\right)^{\beta_m-\alpha_m} + \cdots + \frac{2\alpha_k}{\alpha_{a+1}-\beta_{a+1}} \left(\log_{a+1} \frac{1}{r_0}\right)^{\beta_{a+1}-\alpha_{a+1}} \left(\log_{a+1} \frac{1}{r_0}\right)^{\beta_{a+2}-\alpha_{a+2}-1} \cdots$$

$$\cdots \left(\log_k \frac{1}{r_0}\right)^{\beta_k-\alpha_k-1} \left(\log_{k+1} \frac{1}{r_0}\right)^{\beta_{k+1}-\alpha_{k+1}} \cdots \left(\log_m \frac{1}{r_0}\right)^{\beta_m-\alpha_m} + \cdots +$$

$$+ \frac{2\alpha_m}{\alpha_{a+1}-\beta_{a+1}} \left(\log_{a+1} \frac{1}{r_0}\right)^{\beta_{a+1}-\alpha_{a+1}} \left(\log_{a+2} \frac{1}{r_0}\right)^{\beta_{a+2}-\alpha_{a+2}-1} \times$$

$$\times \left(\log_m \frac{1}{r_0}\right)^{\beta_m-\alpha_m-1} < \infty.$$

Next, for $\alpha_{a+1} = \beta_{a+1}$ and $\alpha_{a+2} > \beta_{a+2} + 1$, we get

$$\int_0^{r_0} \Phi(r) dh(r) = \alpha_{a+1} \int_{\log_{a+2}\frac{1}{r_0}}^{\infty} v^{\beta_{a+2}-\alpha_{a+2}} \cdots (\log_{m-a-2} v)^{\beta_m-\alpha_m} dv + \cdots +$$

$$+ \alpha_m \int_{\log_{a+2}\frac{1}{r_0}}^{\infty} v^{\beta_{a+2}-\alpha_{a+2}-1} \cdots (\log_{m-a-2} v)^{\beta_m-\alpha_m-1} dv \leqq$$

$$\leqq \alpha_{a+1} \left(\log_{a+2} \frac{1}{r_0}\right)^{\frac{\beta_{a+2}-\alpha_{a+2}+1}{2}} \left(\log_{a+3} \frac{1}{r_0}\right)^{\beta_{a+3}-\alpha_{a+3}} \cdots$$

$$\cdots \left(\log_m \frac{1}{r_0}\right)^{\beta_m-\alpha_m} \int_{\log_{a+2}\frac{1}{r_0}}^{\infty} v^{\frac{\beta_{a+2}-\alpha_{a+2}+1}{2}-1} dv + \cdots +$$

$$+ \alpha_k \left(\log_{a+2} \frac{1}{r_0}\right)^{\frac{\beta_{a+2}-\alpha_{a+2}}{2}} \left(\log_{a+2} \frac{1}{r_0}\right)^{\beta_{a+3}-\alpha_{a+3}-1} \cdots$$

$$\cdots \left(\log_k \frac{1}{r_0}\right)^{\beta_k-\alpha_k-1} \left(\log_{k+1} \frac{1}{r_0}\right)^{\beta_{k+1}-\alpha_{k+1}} \cdots$$

$$\cdots \left(\log_m \frac{1}{r_0}\right)^{\beta_m-\alpha_m} \int_{\log_{a+2}\frac{1}{r_0}}^{\infty} v^{\frac{\beta_{a+2}-a_{a+2}}{2}-1} dv +$$

$$+ \cdots + \alpha_m \left(\log_{a+2} \frac{1}{r_0}\right)^{\frac{\beta_{a+2}-\alpha_{a+2}}{2}} \left(\log_{a+3} \frac{1}{r_0}\right)^{\beta_{a+3}-\alpha_{a+3}-1} \cdots$$

$$\cdots \left(\log_m \frac{1}{r_0}\right)^{\beta_m-a_m-1} \int_{\log_{a+2}\frac{1}{r_0}}^{\infty} v^{\frac{\beta_{a+2}-\alpha_{a+2}}{2}-1} dv =$$

$$= \frac{2\alpha_{a+1}}{\alpha_{a+2} - \beta_{a+2} + 1}\left(\log_{a+2}\frac{1}{r_0}\right)^{\beta_{a+2}-\alpha_{a+2}+1}\left(\log_{a+3}\frac{1}{r_0}\right)^{\beta_{a+3}-\alpha_{a+3}} \ldots$$

$$\ldots\left(\log_m\frac{1}{r_0}\right)^{\beta_m-\alpha_m} + \ldots +$$

$$+ \frac{2\alpha_k}{\alpha_{a+2} - \beta_{a+2}}\left(\log_{a+2}\frac{1}{r_0}\right)^{\beta_{a+2}-\alpha_{a+2}}\left(\log_{a+3}\frac{1}{r_0}\right)^{\beta_{a+3}-\alpha_{a+3}-1} \ldots$$

$$\ldots\left(\log_k\frac{1}{r_0}\right)^{\beta_k-\alpha_k-1}\left(\log_{k+1}\frac{1}{r_0}\right)^{\beta_{k+1}-\alpha_{k+1}}\ldots\left(\log_m\frac{1}{r_0}\right)^{\beta_m-\alpha_m} + \ldots +$$

$$+ \frac{2\alpha}{\alpha_{a+2} - \beta_{a+2}}\left(\log_{a+2}\frac{1}{r_0}\right)^{\beta_{a+2}-\alpha_{a+2}}\left(\log_{a+3}\frac{1}{r_0}\right)^{\beta_{a+3}-\alpha_{a+3}-1} \ldots$$

$$\ldots\left(\log_m\frac{1}{r_0}\right)^{\beta_m-\alpha_m-1} < \infty.$$

And finally, in the case $\alpha_{a+1} = \beta_{a+1}$, $\alpha_q = \beta_q + 1$ $(q = a + 2, \ldots, a + s < m)$ and $\alpha_{a+s+1} > \beta_{a+s+1} + 1$, we obtain

$$\int_0^{r_0}\Phi(r)\,dh(r) = \alpha_{a+1}\int_{\log_{a+1}\frac{1}{r_0}}^{\infty}(\log u)^{-1}\ldots(\log_{s-1}u)^{-1}(\log_s u)^{\beta_{a+s+1}-\alpha_{a+s+1}}\ldots$$

$$\ldots(\log_{m-a-1}u)^{\beta_m-\alpha_m}\frac{du}{u} + \alpha_{a+2}\int_{\log_{a+1}\frac{1}{r_0}}^{\infty}(\log u)^{-2}(\log_2 u)^{-1}\ldots$$

$$\ldots(\log_{s-1}u)^{-1}(\log_s u)^{\beta_{a+s+1}-\alpha_{a+s+1}}\ldots(\log_{m+a+1}u)^{\beta_m-\alpha_m}\frac{du}{u} + \ldots +$$

$$+ \alpha_{a+b}\int_{\log_{a+1}\frac{1}{r_0}}^{\infty}(\log u)^{-2}\ldots(\log_b u)^{-2}(\log_{b+1}u)^{-1}\ldots$$

$$\ldots(\log_{s-1}u)^{-1}(\log_s u)^{\beta_{a+s+1}-\alpha_{a+s+1}}\ldots(\log_{m-a-1}u)^{\beta_m-\alpha_m}\frac{du}{u} + \ldots +$$

$$+ \alpha_m\int_{\log_{a+1}\frac{1}{r_0}}^{\infty}(\log u)^{-2}\ldots(\log_{s-1}u)^{-2}(\log_s u)^{\beta_{a+s+1}-\alpha_{a+s+1}-1}\ldots$$

$$\ldots(\log_{m-a-1}u)^{\beta_m-\alpha_m-1}\frac{du}{u} < \alpha_{a+1}\int_{\log_{a+s+1}\frac{1}{r_0}}^{\infty}v^{\beta_{a+s+1}-\alpha_{a+s+1}}\ldots$$

$$\ldots(\log_{m-a-s-1}v)^{\beta_m-\alpha_m} + \ldots + \alpha_m\int_{\log_{a+s}\frac{1}{r_0}}^{\infty}v^{\beta_{a+s+1}-\alpha_{a+s+1}-1}\ldots$$

$$\ldots(\log_{m-a-s-1}v)^{\beta_m-\alpha_m-1}\,dv < \alpha_{a+1}\left(\log_{a+s+1}\frac{1}{r_0}\right)^{\frac{\beta_{a+s+1}-\alpha_{a+s+1}+1}{2}}\cdot$$

$$\cdot\left(\log_{a+s+2}\frac{1}{r_0}\right)^{\beta_{a+s+2}-\alpha_{a+s+2}}\ldots\left(\log_m\frac{1}{r_0}\right)^{\beta_m-\alpha_m}\int_{\log_{a+s+1}\frac{1}{r_0}}^{\infty}v^{\frac{\beta_{a+s+1}-\alpha_{a+s+1}}{2}-1}\,dv + \ldots +$$

$$+ \alpha_m\left(\log_{a+s+1}\frac{1}{r_0}\right)^{\frac{\beta_{a+s+1}-\alpha_{a+s+1}}{2}}\left(\log_{a+s+2}\frac{1}{r_0}\right)^{\beta_{a+s+2}-\alpha_{a+s+2}-1}\ldots$$

$$\ldots\left(\log_m\frac{1}{r_0}\right)^{\beta_m-\alpha_m-1}\int_{\log_{a+s+1}\frac{1}{r_0}}^{\infty}v^{\frac{\beta_{a+s+1}-\alpha_{a+s+1}+1}{2}-1}\,dv =$$

$$= \frac{2\alpha_{a+1}}{\alpha_{a+s+1} - \beta_{a+s+1} + 1}\left(\log_{a+s+1}\frac{1}{r_0}\right)^{\beta_{a+s+1}-\alpha_{a+s+1}+1}\left(\log_{a+s+2}\frac{1}{r_0}\right)^{\beta_{a+s+2}-\alpha_{a+s+2}}\ldots$$

$$\ldots\left(\log_m\frac{1}{r_0}\right)^{\beta_m-\alpha_m} + \ldots + \frac{2\alpha_m}{\alpha_{a+s+1} - \beta_{a+s+1}}\left(\log_{a+s+1}\frac{1}{r_0}\right)^{\beta_{a+s+1}-\alpha_{a+s+1}}\cdot$$

$$\cdot\left(\log_{a+s+2}\frac{1}{r_0}\right)^{\beta_{a+s+2}-\alpha_{a+s+2}-1}\ldots\left(\log_m\frac{1}{r_0}\right)^{\beta_m-\alpha_m-1} < \infty,$$

as desired.

Remarks. 1. S. J. TAYLOR [19] observes that in the relation between Hausdorff h-measures and Φ-capacities C_Φ established by means of the preceding proposition, the interval $\alpha = \mu$, $\nu - 1 \leqq \beta < \nu$ remains as a gap of uncertainty of a multiplying factor $\log\frac{1}{r}$, respectively $\log_2\frac{1}{r}$. In the more general case of the preceding lemma, the corresponding gap of uncertainty is given by the interval $\alpha_{a+1} = \beta_{a+1}$ and $\beta_{a+2} < \alpha_{a+2} \leqq \beta_{a+2} + 1$ (as in the preceding proposition, the factor being $\log_{a+1}\frac{1}{r}$), or $\alpha_{a+1} = \beta_{a+1}$, $\beta_q < \alpha_q < \beta_q + 1$ $(q = a + 2, \ldots, a + s)$ and $\beta_{a+s+1} \leqq \alpha_{a+s+1} \leqq \beta_{a+s+1} + 1$ (the factors being $\log_{a+2}\frac{1}{r}\ldots\log_{a+s+1}\frac{1}{r}$.

2. We wish to mention that in the case of our results on the exceptional sets of Bessel or conformal capacity zero, we did not use only the relation between Hausdorff h-measure and Bessel capacity $B_{\alpha,p}$ or Φ-capacities given by the preceding lemma and (specially by means of lemma 4 and theo-

rem 3) we succeeded to have no more intervals of uncertainty. In this case, according to theorems 4 and 5 and their corollaries, the estimates obtained are showed to be best possible, while the corresponding estimates expressed by means of the different capacities have only points of uncertainty corresponding to $\beta_m = p - 2$.

And now, let us give the following generalization of the conformal capacity belonging to J. SERRIN [17]:

Let E be a bounded set in R^n. $\mathrm{Cap}_p\, E$, where $1 \leqq p < \infty$ is defined by

$$\mathrm{cap}_p\, E = \inf_u \int |\nabla u|^p\, dx,$$

where the infimum is taken over all $u \in C_0^1$ (i.e. continuously differentiable and with compact support) and which are $\geqq 1$ on E. If $p \geqq n$, we also require the support S_u of u to be contained in a certain fixed ball $B(R_0)$, which is independent of E.

Now, let us show the connection between Φ-capacities and p-modules.

Proposition 19. *Suppose $F \subset R^n$ is a compact set, $p \geqq 1$ and Γ_F is the family of all arcs that intersect F. Then* $\mathrm{cap}_p F = 0 \Leftrightarrow M_p(\Gamma_F) = 0$.

(For the proof, see W. ZIEMER [24].)

Proposition 20. *Suppose $F \subset R^n$ is a compact set and $p \geqslant 1$, then* $\mathrm{cap}_p\, F = 0 \Leftrightarrow B_{\alpha, p}(F) = 0$.

(For the proof, see for instance JU. G. REŠETNJAK [16], § 6, or H. WALLIN [22], theorem 1.)

From the preceding 2 propositions and from theorems 4 and 5, we obtain

Corollary 1. *If $F \subset R^n$ is compact, $p \leqslant n$ and* $\mathrm{cap}_p\, F = 0$, *or $M_p(\Gamma_F) = 0$, then $H_{h_{p,\, p-1,\, m,\, \beta}}(F) = 0\ \forall \beta > p - 1$ $(m = 1, 2, \ldots)$ and this result is best possible in the sense that there exist Cantor sets F' with* $\mathrm{cap}_p\, F' = M_p\Gamma_F) = 0$, *but with $0 < H_{h_{p,\, p-1,\, m,\, \beta}}(F') < \infty\ \forall \beta \leqq p - 1$ $(m = 1, 2, \ldots)$.*

From the preceding 2 propositions and from corollary 1 of lemma 14, we deduce

Corollary 2. *If $F \subset R^n$ is a compact set, $2 \leqq p \leqq n$ and* $\mathrm{cap}_p\, F = 0$, *or $M_p(\Gamma_F) = 0$, then $C_{\Phi_{p,\, p-2,\, m,\, \beta}}(F) = 0\ \forall \beta > p - 2$ $(m = 1, 2, \ldots)$, but if $p = 2\ \forall \beta \geqq 0$ and there are Cantor sets F_1 such that* $\mathrm{cap}_p\, F_1 = M_p(\Gamma_F) = 0$, *while $C_{\Phi_{p,\, p-2,\, m,\, \beta}}(F_1) > 0\ \forall \beta < p - 2$ $(m = 1, 2, \ldots)$.*

Remarks. 1. This corollary represents an extension of theorem (B) of H. WALLIN [21] asserting that $\mathrm{cap}_p\, F = 0\ (2 < p \leqq n) \Rightarrow C_{n-p+\varepsilon} F = 0$ $(\forall \varepsilon > 0)$. H. WALLIN [21] in his remark 2 (p. 339) asserts that his result is best possible for $2 < p < n$ in the sense that: „If $2 < p < n$, there exist compact sets F satisfying $\mathrm{cap}_p\, F = 0$ and $C_{n-p} F > 0$”. However, Wallin's result is not best possible „in an absolute sense” since, according to the preceding corollary, we have, for $2 \leqq p < n$,

$$C_{n-p+\varepsilon} \prec C_{\Phi_{p,\, p-2,\, m,\, \beta}} \prec \mathrm{cap}_p \quad \forall \varepsilon > 0.$$

2. In the particular case $n = 2$, the first part of the preceding corollary gives the classical result „cap $F = 0 \Leftrightarrow C_0(F) = 0$”.

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