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A CHARACTERIZATION OF BOUNDARIES OF SMOOTH STRICTLY CONVEX PLANE SETS

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Boundaries of bounded convex sets in the plane have been characterized by K. MENGER [4] (see also F. A. VALENTINE [7], pp. 113–115) by certain simple conditions expressible in terms of the three-point subsets of S. Related results have been obtained by W. M. SWAN [6] and more recently by K. JUUL [1], who extended Menger's theorem to possibly unbounded closed sets.

In this note we characterize the boundaries of smooth strictly convex compact sets in the Euclidean plane $\mathbf{R}^2$ in terms of the three-point subsets and of the existence and unicity of inscribed triangles.

Terminology

Let S be a set in the plane $\mathbf{R}^2$. We shall denote by int S, bd S and conv S the interior, the boundary and respectively the convex hull of the set S. The closed and open segments with endpoints x and y are denoted by $[x, y]$ and $]x, y[$, respectively. If x, y, z are noncollinear points, $L(x, y)$ and $H(x, y;\ z)$ denote the line through x and y, and the closed half-plane H with $x, y \in \text{bd}\, H$, $z \in H$, respectively. A convex body in a linear topological space is said to be *smooth*, if in each boundary point of S there exists only one supporting hyperplane. We say that a convex body S is *strictly convex*, if bd S doesn't contain any segment, or with other words: for $x, y \in S$ and $x \neq y$ we have $]x, y[\subset \text{int}\, S$.

In the sequel we need the following result of K. JUUL [1]:

THEOREM 1. *A plane set S fulfils*

(i) $$\forall x, y, z \in S:\ S \cap \text{int conv}\, \{x, y, z\} = \emptyset$$

if and only if S is either a subset of the boundary of a convex set, or an X-set, that is a set $\{x_1, x_2, x_3, x_4, x_5\}$ *with* $]x_1, x_2[\cap]x_3, x_4[= \{x_5\}$.

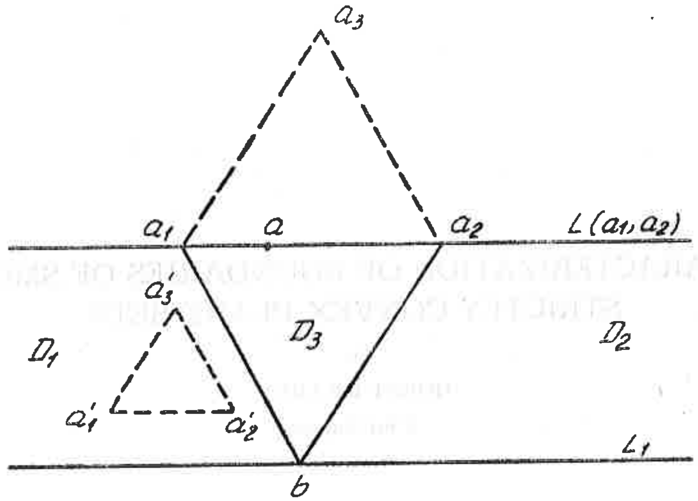

Fig. 1

There is also needed the following theorem of the author and A. B. NÉMETH [2]:

THEOREM 2. *Let abc be a triangle in the Euclidean plane* $\mathbf{R}^2$. *Suppose that S is a strictly convex closed arc of class* C^1. *Then there exists a single triangle* $a_1b_1c_1$ *with sides parallel to sides of abc and of the same orientation as abc and which is inscribed in S, in the sense that* $a_1, b_1, c_1 \in S$.

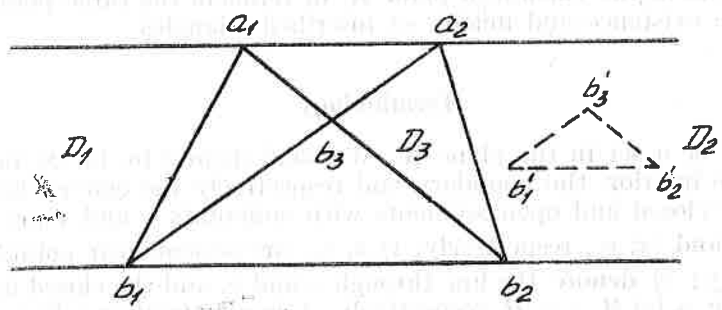

Fig. 2

A generalization of this theorem was given in ([3], Theorem 1).

Results and proofs

THEOREM 3. *A plane compact set S is the boundary of a smooth strictly convex set if and only if the following two conditions hold:*

(i) $\forall x, y, z \in S: \quad S \cap \operatorname{int} \operatorname{conv} \{x, y, z\} = \emptyset$

(ii) *For every triangle* $p_1p_2p_3$ *in* $\mathbf{R}^2$ *there is only one triangle* $p_1' p_2' p_3'$ *with sides parallel to the sides of* $p_1p_2p_3$ *and of the same orientation as* $p_1p_2p_3$ *and which is inscribed in S in the sense that* $p_1', p_2', p_3' \in S$.

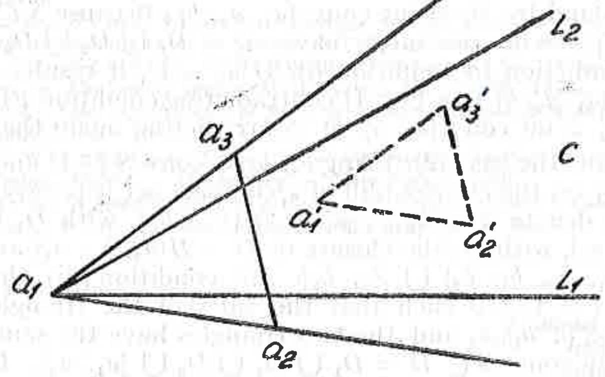

Fig. 3

Proof. The „if" statement follows immediately from Theorem 2 and from the fact that S is the boundary of a strictly convex set.

To prove the „only if" statement, let us suppose that S is a plane compact set fulfilling conditions (i) and (ii). By Theorem 1, S is either a subset of the boundary of a convex set or an X-set. Since S verifies also condition (ii), it cannot be an X-set. That means that $S \subset \operatorname{bd} \operatorname{conv} S$. Suppose now that we have $\operatorname{bd} \operatorname{conv} S \not\subset S$, i.e. there is a point a, $a \notin S$ and $a \in \operatorname{bd} \operatorname{conv} S$. But as the convex hull of a compact set in $\mathbf{R}^n$ is compact (see for instance [5], Theorem 3.2.18) it follows that $a \in \operatorname{bd} \operatorname{conv} S \subset \operatorname{conv} S$. By the theorem of Caratheodory on the convex hull of a compact set in the space $\mathbf{R}^n$, there are points $a_1, \ldots, a_i \in S$, with $i \leq 3$ such that $a \in \operatorname{conv} \{a_1, a_2, \ldots, a_i\}$. If $i = 3$ and $a \in \operatorname{int} \operatorname{conv} \{a_1, a_2, a_3\}$, we would have $a \in \operatorname{int} \operatorname{conv} S$, contradicting the above statement $a \in \operatorname{bd} \operatorname{conv} S$. It follows that $i = 2$ and $a \in \operatorname{conv} \{a_1, a_2\}$. Since S is a compact set and $a \notin S$, we can choose the points a_1 and a_2 such that we have $]a_1, a_2[\cap S = \emptyset$. But then $L(a_1, a_2)$ is the only supporting line through a for conv S. Let b be a point in S such that $d(b, L(a_1, a_2)) = \max_{p \in S} d(p, L(a_1, a_2))$. (If $d(x, y)$ is the distance in $\mathbf{R}^2$ between the points x and y, M is a given set and b a point in $\mathbf{R}^2$, we have by definition $d(b, M) = \inf_{p \in M} d(b, p)$). Denote with L_1 the line through b parallel to $L(a_1, a_2)$. It is immediately that conv S is contained in the closed strip D with the boundary formed by the lines L_1 and $L(a_1, a_2)$. We consider now the the following two cases:

(1) $$L_1 \cap \operatorname{conv} S = \{b\}, \text{ and}$$

(2) $$L_1 \cap \operatorname{conv} S = [b_1, b_2] \ni b.$$

In the first case denote by a_3 the intersection point of the line through a_1 parallel to $L(a_2, b)$ and of the line through a_2 parallel to $L(a_1, b)$. By the condition (ii) there has to be a triangle $a_1' a_2' a_3'$ with sides parallel to those of $a_1a_2a_3$ and of the same orientation as $a_1a_2a_3$, such that $a_1', a_2', a_3' \in S$. Denote by D_1 the closure of $D - H(a_1, b ; a_2)$ and by D_2 the closure of $D-H(a_2, b ; a_1)$and by $D_3 = \text{int conv } \{a_1, a_2, b\}$. Because $S \subset \text{conv } S \subset D$ and $]a_1, a_2[\cap S = \emptyset$, we must have $a_3' \in D_1 \cup D_2 \cup D_3$. If $a_3' \in D_3$ we get a contradiction to condition (i). If $a_3' \in D_1$ it results that we have $a_2' \in \text{int conv } \{a_1', a_2, b\}$ (see Fig. 1) contradicting condition (i). If $a_3' \in D_2$ it follows that $a_1' \in \text{int conv } \{a_1, a_2', b\}$, contradicting again the condition (i).

Consider now the case (2). Using $b_1, b_2 \in \text{conv } S \subset D$ and the theorem of Caratheodory on the convex hull of a compact set it is easy to show that $b_1, b_2 \in S$. We denote with $b_3 = [a_1, b_2] \cap [a_2, b_1]$, with D_1 the closure of $D - H(a_1, b_1 ; a_2)$, with D_2 the closure of $D - H(a_2, b_2 ; a_1)$ and with $D_3 = \text{int conv } \{a_1, a_2, b_1, b_2\} \cup]b_1, b_2[$. By condition (ii) there exist the ponts $b_i' \in S$, $i = 1, 2, 3$ such that the sides of the triangle $b_1' b_2' b_3'$ are parallel to those of $b_1b_2b_3$ and the two triangles have the same orientation. We have again $\text{conv } S \subset D = D_1 \cup D_2 \cup D_3 \cup]a_1, a_2[$. If one of the points b_i' would be in D_3 this would be in contradiction with $S \subset \text{bd conv } S$. For $b_3' \in D_1$ results $b_2' \in \text{int conv } \{b_1', a_2, b_2\}$ and for $b_3' \in D_2$ results $b_1' \in \text{int conv } \{a_1, b_1, b_2'\}$. Thus we are again in contradiction to (i).

Both cases (1) and (2) have us led to a contradiction. Hence $\text{bd conv } S \subset S$ and together with the above result $S \subset \text{bd conv } S$ we get $S = \text{bd conv } S$.

We claim now that conv S is a smooth set. Assume, to the contrary, that there is a point $a_1 \in \text{bd conv } S$, which is not a smooth point i.e. there exist two lines L_1 and L_2 supporting the set conv S at a_1. For $i = 1$ or 2 denote with H_i the closed half-plane determined by the supporting line L_i, which contains the set S. Denote with C the cone $C = H_1 \cap H_2$. Consider now an isosceles triangle $a_1a_2a_3$ with $a_1a_2 = a_1a_3$ and such that the angle $a_2a_1a_3$ has the same bisector as the boundary angle of C and the angle $a_2a_1a_3$ is greater than the boundary angle of C. By the condition (ii) there exist three points $a_i' \in S$, $i = 1,2,3$ such that the sides of the triangle $a_1' a_2' a_3'$ are parallel to those of the triangle $a_1a_2a_3$ and the two triangles have the same orientation (see Fig. 3). But then $a_1' \in \text{int conv } \{a_1, a_2', a_3'\}$ contradicting the condition (i). Thus conv S has to be a smooth set.

It remains to show that conv S is also a strictly convex set. Suppose the contrary i.e. there is a line segment $[b_1, b_2]$ contained in $\text{bd conv } S = S$. Consider on the segment $[b_1, b_2]$ the two points a_1 und a_2 such that $b_1a_1 = a_1a_2 = a_2b_2$. Let a_3 be a point of S such that $a_1a_2a_3$ is a nondegenerated triangle. As $S = \text{bd conv } S$ we can find a point $a_3' \in S$ sufficiently near to a_3, $a_3' \neq a_3$, such that the parallel to a_3a_1 through a_3' intersects $]b_1, b_2[$ in a point a_1' and the parallel to a_3a_2 through a_3' intersects $]b_1, b_2[$ in a point a_2'. Of course, the two triangles $a_1a_2a_3$ and $a_1' a_2' a_3'$ have parallel sides and are of the same orientation and both are inscribed in S. This contradicts the unicity part of the condition (ii). With this the proof of Theorem 3 is complete.

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