MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 7, N° 1, 1978, pp. 67–79

ON BILINEAR PROGRAMMING

by

I. MARUŞCIAC
(Cluj-Napoca)

1. Introduction

The general bilinear programming problem can be formulated as follows:

(1) $$\text{maximize } \{f(\mathbf{x},\ \mathbf{y}) = \mathbf{cx} + \mathbf{dy} + \mathbf{x}^T\mathbf{Cy}\}$$

subject to linear constrants

(2) $$\mathbf{Ax} = \mathbf{a},\ \mathbf{x} \geqslant 0$$

(3) $$\mathbf{By} = \mathbf{b},\ \mathbf{y} \geqslant 0,$$

where **A**, **B**, **C** are $r \times m$, $s \times n$, $m \times n$ — matrices respectively and **a**, **b**, **c**, **d**, **x**, **y** are vectors of the appropriate dimension.

Bilinear programming is a generalization of the linear programming that for the first time was formulated in 1968 by ALTMAN, M. [1]. He gave an optimality criterion for the bilinear programming which then gave a tool to construct algorithms to find the local optimum in a finite number of steps. SOKIRJANSKAJA, E. [2] has remarked that one of the Altman's criterion is only a sufficient condition for the optimality and not necessary. She gave then an improvement for this criterion and constructs a finite algorithm for the local maximum of the bilinear programming.

An interesting and comprehensive study on bilinear programming is also done in [3] by VANDAL, A.

In the present paper a simplex-like technique is used to establish simple optimality criteria for the general bilinear programming problems. Then a simplex-like algorithm is described to find a local and global maximum of the problem respectively.

2. Jordan elimination in bilinear programming

Now we shall specify the characteristics of a Jordan elimination step in a bilinear programming.

Thus, let us consider

(4) $$f(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} c_i x_i + \alpha + \sum_{j=1}^{n} d_j y_j + \beta + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_i y_j,$$

(5) $$z_k = \sum_{i=1}^{m} a_{ki} x_i + a_k, \quad k = 1, 2, \ldots, r,$$

(6) $$u_h = \sum_{j=1}^{n} b_{hj} y_j + b_h, \quad h = 1, 2, \ldots, s,$$

and let $b_{pq} \neq 0$ be the pivot element. Then after substituting

$$y_q = \frac{1}{b_{pq}} \Big(- \sum_{j \neq q} b_{pj} y_j + u_p - b_p \Big)$$

in (4), we obtan

$$f(\mathbf{x}, \mathbf{y}; \mathbf{u}) = \sum_{i=1}^{m} c_i x_i + \sum_{j \neq q} d'_j y_j + d'_q u_p + \beta' + \sum_{i=1}^{m} x_i \Big(\sum_{j \neq q} c'_{ij} y_j + c'_{iq} u_p + \delta'_i\Big) =$$

$$= \sum_{i=1}^{m} (c_i + \delta'_i) x_i + \alpha + \sum_{j \neq q} d'_j y_j + d'_q u_p + \beta' + \sum_{i=1}^{m} x_i \Big(\sum_{j \neq q} c'_{ij} y_j + c'_{iq} u_p\Big),$$

where

$$d'_j = (d_j b_{pq} - d_q b_{pj})/b_{pq}, \quad j \neq q,$$

(7) $$c'_{ij} = (c_{ij} b_{pq} - c_{iq} b_{pj})/b_{pq}, \quad j \neq q, \ i = 1, 2, \ldots, m,$$

$$c'_{iq} = c_{iq}/b_{pq}, \quad i = 1, 2, \ldots, m,$$

$$\beta' = (\beta b_{pq} - d_q b_p)/b_{pq},$$

$$\delta'_i = - c_{iq}/b_{pq}, \quad i = 1, 2, \ldots, m,$$

$$d'_q = d_q/b_{pq}.$$

If we consider the simplex tableau

(8)

	$\mathbf{x}$	$\mathbf{y}$	1
$\mathbf{z} =$	$\mathbf{A}$	$\mathbf{0}$	$\mathbf{a}$
$\mathbf{u} =$	$\mathbf{0}$	$\mathbf{B}$	$\mathbf{b}$
$f =$	$\mathbf{e}$	$\mathbf{0}$	α
	$\mathbf{0}$	$\mathbf{d}$	β
	$\mathbf{0}$	$\mathbf{C}$	0

then afler a Jordan elimination step we get the tableau

(9)

	$\mathbf{x}$	$y_1 \ldots u_p \ldots y_n$	1
$z =$	$\mathbf{A}$	$\mathbf{0}$	$\mathbf{a}$
$u_1 =$			
$\vdots$			
$y_q =$	$\mathbf{0}$	$\mathbf{B}'$	$\mathbf{b}'$
$\vdots$			
$u_s =$			
$f =$	$\mathbf{e}$	$\mathbf{0}$	α
	$\mathbf{0}$	$\mathbf{d}'$	β'
	$\mathbf{0}$	$\mathbf{C}'$	δ'

where $\mathbf{C}'$, $\mathbf{b}'$, $\mathbf{d}'$, β', δ' are formed by the elements given in (7) and $\mathbf{B}'$ is the matrix obtained from $\mathbf{B}$ after a standard Jordan step.

From (7)—(9) it is seen that in bilinear programming a Jordan elimination step should be carried out according to the usual rules to which one adds:

Additional rule: if the pivot element is an element of the matrix $\mathbf{B}$ then

$$\mathbf{e}' = \mathbf{e} + \delta'$$

Remark 1. If we take a pivot element in the matrix **A** then instead of tableau (8) we consider the tableau

(8)

	$\mathbf{x}$	$\mathbf{y}$	1
$\mathbf{z} =$	$\mathbf{A}$	$\mathbf{0}$	$\mathbf{a}$
$\mathbf{u} =$	$\mathbf{0}$	$\mathbf{B}$	$\mathbf{b}$
$f =$	$\mathbf{c}$	$\mathbf{0}$	α
	$\mathbf{0}$	$\mathbf{d}$	β
	C^T	$\mathbf{0}$	0

and after a Jordan elimination step with pivot element $a_{pq} \neq 0$ we get the tableau

(9)

	$x_1 \ldots z_p \ldots x_m$	$\mathbf{y}$	1
$z_1, \ldots, x_q =, \ldots, z_r$	$\mathbf{A}'$	$\mathbf{0}$	$\mathbf{a}'$
$u =$	$\mathbf{0}$	$\mathbf{B}$	$\mathbf{b}$
$f =$	$\mathbf{c}'$	$\mathbf{0}$	α'
	$\mathbf{0}$	$\mathbf{d}$	β
	C'^T	0	γ'

Now the additional rule consists in:

$$\mathbf{d}' = \mathbf{d} + \gamma'.$$

3. Optimality criteria

A pair $(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^m \times \mathbf{R}^n$ is called basic feasible solution of the bilinear programming (1) — (3) if $\mathbf{x}$ and $\mathbf{y}$ are basic feasible solution of (2) and (3) respectively.

The following theorem results immediately from the theory of linear programming (see [1]):

THEOREM 1. *If* $(\mathbf{x}^*, \mathbf{y}^*)$ *is an optimal solution of the bilinear programming problem* (1)—(3), *then there is a basic feasible one.*

To obtain a b.f.s. we shall use the Jordan elimination steps, described at the section 2. To simplify the notation we assume that **A** and **B** are of the full rank. Then starting from the tableau

(10)

	$-\mathbf{x}$	$-\mathbf{y}$	1
$\mathbf{0} =$	$\mathbf{A}$	$\mathbf{0}$	$\mathbf{a}$
$\mathbf{0} =$	$\mathbf{0}$	$\mathbf{B}$	$\mathbf{b}$
$f =$	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{0}$	$-\mathbf{d}$	0
	C^T	C	0

and assuming (without loss of generality) that the pivot elements were taken from the first r and s column of **A** and **B** respectively, then after $r + s$ J.s. (Jordan elimination steps) we get the tableau

(11)

	$-x_{r+1} \ldots -x_m$	$-y_{s+1} \ldots -y_n$	1
$x_1, \ldots, x_r =$	$\mathbf{A}_1$	$\mathbf{0}$	$\mathbf{a}^1$
$y_1, \ldots, y_s =$	$\mathbf{0}$	$\mathbf{B}_1$	$\mathbf{b}^1$
$f =$	$\mathbf{p}$	$\mathbf{0}$	P
	$\mathbf{0}$	$\mathbf{q}$	Q
	C_1^T	C_1	0

Lemma 1. *If in* (11) $\mathbf{p} > \mathbf{0}$, $\mathbf{q} > \mathbf{0}$, *then* b.f.s. $(\mathbf{x}^0, \mathbf{y}^0)$, *where* $\mathbf{x}^0 = (\mathbf{a}^1, \mathbf{0})$, $\mathbf{y}^0 = (\mathbf{b}^1, \mathbf{0})$, *is a local maximum of the bilinear programming* (1)—(3).

Proof. From (11) it is seen that $f(\mathbf{x}^0, \mathbf{x}^0) = P + Q$ and

$$f(\mathbf{x}, \mathbf{y}) = -\sum_{i=r+1} p_i x_i + P - \sum_{j=s+1} q_j y_j + Q + \sum_{i=r+1} \sum_{j=s+1} c^1_{ij} x_i y_j.$$

Therefore

$$(12)\quad \begin{aligned} f(\mathbf{x}, \mathbf{y}) - f(\mathbf{x}^0, \mathbf{y}^0) &= \sum_{i=r+1} \sum_{j=s+1} c_{ij}^1 x_i y_j - \sum_{i=r+1} p_i x_i - \sum_{j=s+1} q_j y_j = \\ &= \frac{1}{2} \sum_{i=r+1} x_i \Big(\sum_{j=s+1} c_{ij}^1 y_j - 2p_i \Big) + \frac{1}{2} \sum_{j=s+1} y_j \Big(\sum_{i=r+1} c_{ij}^1 x_i - 2q_j \Big). \end{aligned}$$

Now, if $\mathbf{p} > \mathbf{0}$, $\mathbf{q} > \mathbf{0}$, then from (12) it follows that

$$f(\mathbf{x}, \mathbf{y}) - f(\mathbf{x}^0, \mathbf{y}^0) \leqslant 0,$$

for each $x_i \geqslant 0$, $i = r + 1, \ldots, m$, and $y_j \geqslant 0$, $j = s + 1, \ldots, n$, sufficiently small, i.e. $(\mathbf{x}^0, \mathbf{y}^0)$ is a local maximum for f in $\Omega = \mathbf{X} \times \mathbf{Y}$ where,

$$\mathbf{X} = \{\mathbf{x} \in \mathbf{R}^m | \mathbf{A}\mathbf{x} = \mathbf{a},\ \mathbf{x} \geqslant \mathbf{0}\},\ \mathbf{Y} = \{\mathbf{y} \in \mathbf{R}^n | \mathbf{B}\mathbf{y} = \mathbf{b},\ \mathbf{y} \geqslant \mathbf{0}\}.$$

THEOREM 2. *Let $(\mathbf{x}^0, \mathbf{y}^0)$, $\mathbf{x}^0 = (\mathbf{a}^1, \mathbf{0})$, $\mathbf{y}^0 = (\mathbf{b}^1, \mathbf{0})$ be a nondegenerate b.f.s. then $(\mathbf{x}^0, \mathbf{y}^0)$ is a local maximum of f on Ω if and only if*

(i) $\mathbf{p} \geqslant \mathbf{0}$, $\mathbf{q} \geqslant \mathbf{0}$

(ii) $c_{ij}^1 \leqslant 0$, $\forall (i, j) \in I^0 \times J^0$,

where

$$I = \{r + 1, \ldots, m\},\ J = \{s + 1, \ldots, n\}$$

$$I^0 = \{i \in I | p_i = 0\},\ J^0 = \{J \in J | q_j = 0\}.$$

Proof. ($\Leftarrow$). From (11) we have

$$(13)\quad \begin{aligned} f(\mathbf{x}, \mathbf{y}) - f(\mathbf{x}^0, \mathbf{y}^0) &= -\sum_{i \in I} p_i x_i - \sum_{j \in J} q_j y_j + \sum_{i \in I} \sum_{j \in J} c_{ij}^1 x_i y_j = \\ &= \sum_{i \notin I^0} \Big(\sum_{j \in J} c_{ij}^1 y_j - p_i \Big) x_i + \sum_{j \notin J^0} \Big(\sum_{i \in I} c_{ij}^1 x_i - q_j \Big) y_j + \sum_{i \in I^0} \Big(\sum_{i \in J^0} c_{ij}^1 y_j \Big) x_j \end{aligned}$$

From (13) it is clear that (i)—(ii) implies that

$$f(\mathbf{x}, \mathbf{y}) - f(\mathbf{x}^0, \mathbf{y}^0) \leqslant 0$$

for each $x_i \geqslant 0$, $y_j \geqslant 0$ sufficiently small, i.e. $(\mathbf{x}^0, \mathbf{y}^0)$ is a local maximum.

($\Rightarrow$) Let $(\mathbf{x}^0, \mathbf{y}^0)$ be a local maximum and consider

$$\mathbf{x}^i = (\mathbf{a}^1, 0, \ldots, x_i, \ldots, 0)^T,\ x_i = t > 0,\ i \in I$$

$$\mathbf{y}^j = (\mathbf{b}^1, 0, \ldots, y_j, \ldots, 0)^T\ y_j = t > 0,\ j \in J.$$

It is easy to see from (13) that

$$(14)\quad f(\mathbf{x}^i, \mathbf{y}^j) - f(\mathbf{x}^0, \mathbf{y}^0) = \begin{cases} c_{ij}^1 t^2, & i \in I^0,\ j \in J^0 \\ (c_{ij}^1 t - q_j)t, & i \in I^0,\ j \notin J^0 \\ (c_{ij}^1 t - p_i)t, & i \notin I^0,\ j \in J \end{cases}$$

and so $f(\mathbf{x}^i, \mathbf{y}^j) - f(\mathbf{x}^0, \mathbf{y}^0) \leqslant 0$ for $x_i > 0$, $y_j > 0$ sufficiently small, implies (i)—(ii).

Now, let $(\mathbf{x}^0, \mathbf{y}^0)$ be a degenerate b.f.s. $\mathbf{x}^0 = (\mathbf{a}^1, \mathbf{0})$, $\mathbf{y}^0 = (\mathbf{b}^1, \mathbf{0})$, and let us denote

$$I_d = \{i \in \{1, 2, \ldots, r\} \mid a_i^1 = 0\}$$

$$J_d = \{j \in \{1, 2, \ldots, s\} \mid b_j^1 = 0\}$$

THEOREM 3. *Degenerate b.f.s. $(\mathbf{x}^0, \mathbf{y}^0)$ is a local maximum of f on Ω if and only if*

(i) $\mathbf{p} \geqslant \mathbf{0}$, $\mathbf{q} \geqslant \mathbf{0}$

(ii)$_d$ $c_{ij}^1 \leqslant 0$, $\forall (i, j) \in I_d^0 \times J_d^0$

where

$$I_d^0 = \{i \in I^0 \mid a_{ik}^1 < 0,\ \forall k \in I_d\}$$

$$J_d^0 = \{j \in J^0 \mid b_{jk}^1 < 0,\ \forall k \in J_d\}$$

Proof. ($\Rightarrow$) If $(\mathbf{x}^0, \mathbf{y}^0)$ is a b.f.s. that is a local maximum, then

$$f(\mathbf{x}, \mathbf{y}) - f(\mathbf{x}^0, \mathbf{y}^0) \leqslant 0$$

in a certain neighborhood of $(\mathbf{x}^0, \mathbf{y}^0)$. Considering

$$\mathbf{x}^i = (\mathbf{a}^1, 0, \ldots, x_i, \ldots, 0),\ x_i = t > 0,\ i \in I,$$

we have

$$f(\mathbf{x}^i, \mathbf{y}^0) - f(\mathbf{x}^0, \mathbf{y}^0) = \begin{cases} 0, & i \in I^0 \\ -p_i t, & i \notin I^0 \end{cases}$$

As $f(\mathbf{x}^i, \mathbf{y}^0) - f(\mathbf{x}^0, \mathbf{y}^0) \leqslant 0$, it follows that $p_i > 0$, $i \notin I^0$, i.e. $\mathbf{p} \geqslant \mathbf{0}$.

Since for $\mathbf{x}^i \in \mathbf{X}$ it is necessary that

$$\forall k \in I_d \Rightarrow a_{ik}^1 \leqslant 0$$

Similarly we get $\mathbf{q} \geqslant \mathbf{0}$ and

$$\forall k \in J_d \Rightarrow b_{ik}^1 \leqslant 0,\ j \in J.$$

From (14) it is seen that

$$f(\mathbf{x}^i, \mathbf{y}^j) - f(\mathbf{x}^0, \mathbf{y}^0) \leqslant 0$$

implies

$$c_{ij}^1 \leqslant 0,\ \forall (i, j) \in I_d^0 \times J_d^0$$

i.e. (i)—(ii) hold.

($\Leftarrow$). Lemma 2. *Let* $(\mathbf{x}^0, \mathbf{y}^0)$ *be a degenerate b.f.s. If there is* $a^1_{i\nu} > 0$, $i \in I_d$, *or there is* $b^1_{j\mu} > 0$, $j \in J_d$, *then for every* $t > 0$, $(\mathbf{x}^\nu, \mathbf{y}^\mu)$ *is not feasible solution, where*

$$\mathbf{x}^\nu = (\mathbf{a}^1, 0, \ldots, x_\nu, \ldots, 0)^T, \quad x_\nu = t > 0, \tag{15}$$
$$\mathbf{y}^\mu = (\mathbf{b}^1, 0, \ldots, y_\mu, \ldots, 0)^T, \quad y_\mu = t > 0.$$

Proof. Consider $\mathbf{x}^\nu$ defined in (15) and let $a^1_{i\nu} > 0$, $i \in I_d$, then $x_i = -a^1_{i\nu} t < 0$, $\forall t > 0$, that means $\mathbf{x}^\nu \notin \mathbf{X}$, $\forall t > 0$.

Similarly it can be proved that $\mathbf{y}^\mu \notin \mathbf{Y}$, $\forall t > 0$.

Now the sufficiency of the conditions (i)–$(\text{ii})_d$ follows directly from Lemma 2 and (14).

5. Global maximum of the bilinear programming

Assume that $(\mathbf{x}^0, \mathbf{y}^0)$, $\mathbf{x}^0 = (\mathbf{a}^1, \mathbf{0})$ is a local maximum of f on Ω. Then it follows that (i)–(ii) or (i)–$(\text{ii})_d$ hold. For $\mathbf{x}^i = (\mathbf{a}^1, 0, \ldots, x_i, \ldots, 0)$, $\mathbf{y}^j = (\mathbf{b}^1, 0, \ldots, y_j, \ldots, 0)$ we have

$$f(\mathbf{x}^i, \mathbf{y}^j) = -p_i x_i + P - q_j y_j + Q + c^1_{ij} x_i y_j,$$

$$\frac{\partial f(\mathbf{x}^i, \mathbf{y}^j)}{\partial x_i} = -p_i + c^1_{ij} y_j,$$

$$\frac{\partial f(\mathbf{x}^i, \mathbf{y}^j)}{\partial y_j} = -q_j + c^1_{ij} x_i.$$

Conditions (i) imply

$$\frac{\partial f(\mathbf{x}^0, \mathbf{y}^0)}{\partial x_i} \leq 0, \; i \in I, \quad \frac{\partial f(\mathbf{x}^0, \mathbf{y}^0)}{\partial y_j} \leq 0, \; j \in J$$

Define

$$x_i^* = \min_j \{t > 0 \mid f(\mathbf{x}^i, \mathbf{y}^j) - f(\mathbf{x}^0, \mathbf{y}^0) = 0\}, \tag{16}$$
$$y_j^* = \min_i \{t > 0 \mid f(\mathbf{x}^i, \mathbf{y}^j) - f(\mathbf{x}^0, \mathbf{y}^0) = 0\},$$

where $x_i = y_j = t$.

If there is $j \in J$ such that $f(\mathbf{x}^i, \mathbf{y}^j) - f(\mathbf{x}^0, \mathbf{y}^0) = 0$ has no positive solution, then one takes $x_i^* = +\infty$. Similarly, if there is $i \in I$ such that $f(\mathbf{x}^i, \mathbf{y}^j) - f(\mathbf{x}^0, \mathbf{y}^0) = 0$ has no positive solution then $y_j^* = +\infty$.

Consider the inequalities

$$\sum_{i \in I} \frac{x_i}{x_i^*} \leq 1 \tag{17}$$
$$\sum_{j \in J} \frac{y_j}{y_j^*} \leq 1^{*)}$$

Since function f touch its maximum in a b.f.s., it follows that

$$\max \{f(\mathbf{x}, \mathbf{y}) \mid (\mathbf{x}, \mathbf{y}) \in \mathbf{X}^1 \times \mathbf{Y}^1\} = f(\mathbf{x}^0, \mathbf{y}^0),$$

where

$$\mathbf{X}^1 = \mathbf{X} \cap \left\{x \in \mathbf{R}^m \mid \sum_{i \in I} \frac{x_i}{x_i^*} < 1\right\},$$

$$\mathbf{Y}^1 = \mathbf{Y} \cap \left\{y \in \mathbf{R}^n \mid \sum_{j \in J} \frac{y_j}{y_j^*} < 1\right\}.$$

That is why, in order to determine a new local maximum, we shall find a local maximum of f on $(\mathbf{X} \setminus \mathbf{X}^1) \times (\mathbf{Y} \setminus \mathbf{Y}^1)$, i.e. adding to the initial constraints (2)–(3) the following two:

$$\sum_{i \in I} \frac{x_i}{x_i^*} \geq 1; \quad \sum_{j \in J} \frac{y_j}{y_j^*} \geq 1. \tag{18}$$

With this new bilinear programming we proceed similar until the problem becomes inconsistent.

6. Description of the algorithm

From above we conclude with the following algorithm for the global maximum of the bilinear programming problem.

Step 1. Starting from the tableau (10) find a b.f.s.

Step 2. Tests the condition (i). If (i) holds then go to step 3, otherwise do a J.s. by pivot element chosen in a column for which $p_i < 0$ or $q_j < 0$.

Step 3. Tests (ii) or $(\text{ii})_d$. If (ii) or $(\text{ii})_d$ hold then go to Step 4. Otherwise, if

$$c^1_{ij} > 0, \; (i, j) \in \mathrm{I}^0_d \times \mathrm{J}^0_d$$

do two. J.s. by chosing the pivot elements in the column i and j respectively and go to Step 2.

---

*) The terms corresponding to $x_i^* = +$ or $y_j^* = +$ are missing in (17).

Step 4. Determine x_i^* and y_j^* from (16).

Step 5. Add the inequalities (17) to the initial constraints and go to Step 1.

The algorithm is terminated when the bilinear programming problem is inconsistent.

7. Example

To illustrate the algorithm we solve the following example: maximize

$$f(\mathbf{x}, \mathbf{y}) = -x_1 - 11x_2 - 8y_1 - 4y_2 + 2x_1y_1 - x_1y_2 + 6x_2y_1 + 5x_2y_2$$

subject to

$$\begin{aligned} x_1 \quad + x_3 \quad &= 2 \\ x_2 + \quad x_4 &= 2 \\ y_1 \quad + y_3 \quad &= 2 \\ y_2 \quad + y_4 &= 2 \end{aligned}$$

$$x_i \geqslant 0,\ y_j \geqslant 0,\ i, j = 1, 2, 3, 4.$$

Step 1. The initial tableau is:

	$-x_1$	$-x_2$	$-x_3$	$-x_4$	$-y_1$	$-y_2$	$-y_3$	$-y_4$	1
0 =	1	0	1	0	0	0	0	0	2
0 =	0	1	0	1	0	0	0	0	2
0 =	0	0	0	0	[1]	0	1	0	2
0 =	0	0	0	0	0	1	0	1	2
	1	11	0	0	0	0	0	0	0
	0	0	0	0	8	4	0	0	0
$f =$					2	−1	0	0	0
					6	5	0	0	0
					0	0	0	0	0
					0	0	0	0	0

After a J. s. we got

	$-x_1$	$-x_2$	$-x_3$	$-x_4$	$-y_2$	$-y_3$	$-y_4$	1
0 =	1	0	1	0	0	0	0	2
0 =	0	1	0	1	0	0	0	2
$y_1 =$	0	0	0	0	0	1	0	2
0 =	0	0	0	0	1	0	1	2
	1	11	0	0	0	0	0	0
	0	0	0	0	4	−8	0	−16
$f =$					−1	−2	0	− 4
					5	−6	0	−12
					0	0	0	0
					0	0	0	0

After other three J.s. we get the tableau

	$-x_2$	$-x_3$	$-y_2$	$-y_3$	1
$x_1 =$	0	1	0	0	2
$x_4 =$	1	0	0	0	2
$y_1 =$	0	0	0	1	2
$y_4 =$	0	0	1	0	2
	−1	3	0	0	6
	0	0	4	−8	−16
$f =$	5	1			2
	−6	2			4

Step 2. Since (i) do not hold (there is −1 and −4 = −8 + 4 negative elements) we do other two J.s. and we get the tableau

	$-x_4$	$-x_3$	$-y_4$	$-y_3$	1
$x_1 =$	0	1	0	0	2
$x_2 =$	1	0	0	0	2
$y_1 =$	0	0	0	1	2
$y_2 =$	0	0	1	0	2
	11	1	0	0	8
$f =$	0	0	4	8	−8
			5	6	2
			−1	2	−0

Step 3. Since $p_4 = 11$, $p_3 = 1$, $q_4 = 4$, $q_3 = 8$ it follows that (x^0, y^0), where

$$\mathbf{x}^0 = (2, 2, 0, 0)\,;\ \mathbf{y}^0 = (2, 2, 0, 0),$$

is a local maximum and $f(\mathbf{x}^0, \mathbf{y}^0) = 8 - 8 = 0.$

Step 4. We have

$$f(x^4, y^4) - f(\mathbf{x}^0, \mathbf{y}^0) = 5t(t-3) = 0, \text{ i.e. } t_{44} = 3$$

$$f(\mathbf{x}^4, \mathbf{y}^3) - f(\mathbf{x}^0, \mathbf{y}^0) = -19t + 6t^2 = 0,\ t_{34} = 19/6,$$

$$x_4^* = \min\,\{3, 19/6\} = 3.$$

Since

$$f(\mathbf{x}^3, \mathbf{y}^4) - f(\mathbf{x}^0, \mathbf{y}^0) = -5t - t^2 = 0,\quad t_{34} = -5,\quad x_3^* = +\infty.$$

Similarly we obtain

$$y_3^* = 19/6, \text{ and } y_4^* = +\infty.$$

Therefore the inequalities (17) are

$$x_4 \geqslant 3, \text{ and } 6y \geqslant 19.$$

Step 5. The new simplex tableau is the following

	$-x_4$	$-x_3$	$-y_4$	$-y_3$	1
$x_1 =$	0	1	0	0	2
$x_2 =$	1	0	0	0	2
$x_5 =$	−1	0	0	0	−3
$y_1 =$	0	0	0	1	2
$y_2 =$	0	0	1	0	2
$y_5 =$	0	0	0	−6	−19
$f =$	11	1	0	0	8
	0	0	4	8	−8
	5	−1			
	6	2			

Step 1. After one J.s. we get the tableau

	$-x_5$	$-x_3$	$-y_4$	$-y_3$	1
$x_1 =$	0	1	0	0	2
$x_2 =$	1	0	0	0	−1
$x_4 =$	−1	0	0	0	3
$y_1 =$	0	0	0	1	2
$y_2 =$	0	0	1	0	2
$y_5 =$	0	0	0	−6	−19
$f =$	11	1	0	0	−25
	0	0	−11	−10	−8
	5	−1			
	6	2			

which shows that the new problem is inconsistent $(x_2 = -x_5 - 1 < 0$, for every $x_5 \geqslant 0)$.

Therefore $\mathbf{x}^0 = (2, 2, 0, 0)\,;\ \mathbf{y}^0 = (2, 2, 0, 0)$ is the optimal solution and $f(\mathbf{x}^0, \mathbf{y}^0) = 0$.

Univ. „Babeş-Bolyai", Cluj-Napoca

R E F E R E N C E S

[1] Altman, M., *Bilinear programming*. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **16**, 741–746 (1968).

[2] Sokirjanskaia, E. N., *Zamečanija k stat'e M. Al'tmana „Bilineinoe programmirovanie"*. Optimizacija, vyp. **16 (33)**, 91–98 (1975).

[3] Vandal, A., *Bilinear programming*. Ekonomska Analiza 4, Nr. 1–2, 21–41 (1970).

Received 30. III. 1978.

Universitatea Babeş-Bolyai
Facultatea de Matematică
Str. Kogălniceanu 1
3400 *Cluj-Napoca*