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THE NUMERICAL SOLUTION OF THE SELF-ADJOINT SECOND ORDER ELLIPTIC PARTIAL DIFFERENTIAL EQUATION UNDER MIXED BOUNDARY CONDITIONS

by

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Summary. In this paper we present a method for the numerical solution of a model problem of the two dimentional Self-Adjoint second order elliptic partial differential Equation under mixed boundary conditions.

1. Introduction

In a previous paper of ours [3] the first three boundary value problems for the two dimensional self-adjoint second order elliptic partial differential equation (S.A.E.) were studied under the assumption that certain conditions were fulfilled so that Extrapolated Alternating Direction Implicit (E.A.D.I.) methods could be used for the numerical solution of them.

The purpose of this paper is to show how to develop a theory analogous to that in [3] so that the numerical solution of the S.A.E. under mixed boundary conditions could be obtained. As we shall see in the analysis which follows, from the numerical point of view, the solution of the problem in question, presents no difficulties. This agrees wit what is already known (see e.g. GREENSPAN [4] pp. 57—58) and becomes clear by working out a numerical example.

2. Statement of the problem and notation used

To facilitate the subsequent analyisis the following notations are introduced and used throughout this paper

$I \equiv \{1, 2\}$ with $i \in I$ $\qquad \partial R_i = \partial R_i^0 \cup \partial R_i^1$

$J \equiv \{0, 1\}$ with $j \in J$ $\qquad \partial R = \partial R_1 \cup \partial R_2$

$R \equiv \{x | x \equiv (x_1, x_2)$ and $0 < x_i < l_i\}$ $\qquad \bar{R} = R \cup \partial R$

$\partial R_i^j \equiv \{x | x \equiv (x_1, x_2)$ and $x_i = jl_i\}$

Having introduced the notations above we consider now the S.A.E.

(1) $$Lu = f(x), \quad x \in R$$

where L is the elliptic operator defined by

(2) $$L \equiv \frac{\partial}{\partial x_1}\left(\alpha_1(x_1)\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\alpha_2(x_2)\frac{\partial}{\partial x_2}\right) - c_1(x_1) - c_2(x_2)$$

with $\alpha_i(x_i) > 0$ and $c_i(x_i) \geqq 0 | i \in I$. The solution u, which is required to satisfy the boundary conditions

(3) $$lu = v_i^j(x), \quad x \in \partial R_i^j, \quad i \in I \text{ and } j \in J$$

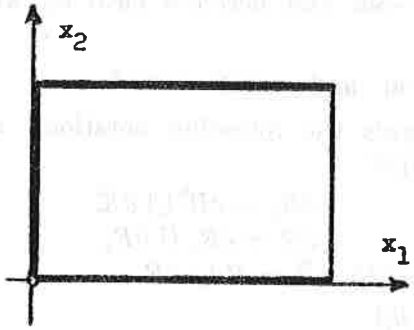
is assumed to be sufficiently smooth in $\bar{R}$. The operator l in (3) is defined as the identity operator on one or more sides ∂R_j^i of the rectangle R and as the operator $(-1)^{j+1}\frac{\partial}{\partial x_i} + \sigma_i^j$ $(\sigma_i^j \geqq 0)$ on the remaining sides ∂R_i^j i.e.

$$l \equiv \begin{cases} \begin{cases} \equiv \text{the identity operator on } \partial R_D \equiv \\ \equiv \partial R_1^0 \cup \partial R_2 \text{ or } \partial R_1^0 \cup \partial R_2^1 \text{ or } R_1 \text{ or } \partial R_1^0 \end{cases} \\ \equiv l_i^j \equiv (-1)^{j+1}\frac{\partial}{\partial x_i} + \sigma_i^j \text{ on } \partial R_N \equiv \partial R - \partial R_D \end{cases}$$

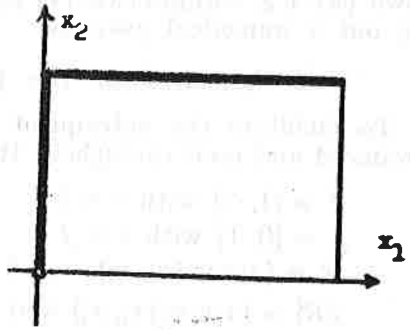
Depending on the way the operator l is defined we can distinguish four different problems. These four problems, which are studied in the sequel, are presented below schematically by giving in bold faced lines the part of the boundary on which the operator l coincides with the identity operator.

We note that in the case where in at least one of the sides of the rectangle R, parallel to the x_i-axis, the boundary conditions are those of a third type boundary value problem the corresponding coeficient $\alpha_i(x_i)$

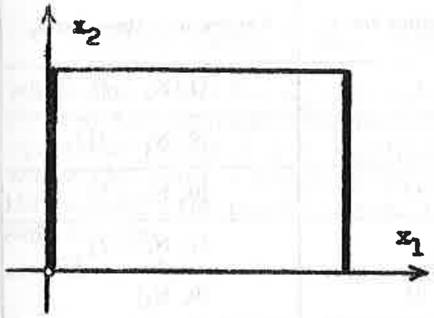
Problem One (PI)
$(\partial R_D = \partial R_1^0 \cup \partial R_2)$



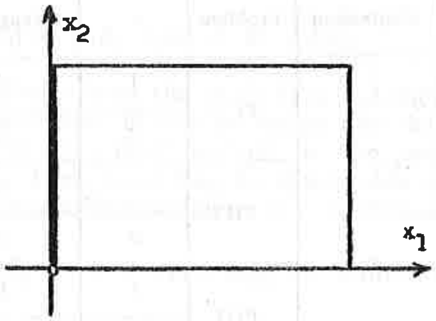
Problem Two (PII)
$(\partial R_D = \partial R_1^0 \cup \partial R_2^1)$



Problem Three (PIII)
$(\partial R_D = \partial R_1)$



Problem Four (PIV)
$(\partial R_D = \partial R_1^0)$



in the elliptic operator L in (2) must be constant is order for the analysis given in this paper to apply (see also [3]).

For the numerical solution of problem (1) — (3) a uniform mesh of size $h_i = l_i/N_i$ in x_i-direction is imposed on $\bar{R}$ where $N_i (\geqq 3)$ is an arbitrarily chosen integer and then the differential equation (1) together with its boundary conditions (3) is approximated by appropriate difference equations at all mesh points. In what follows we adopt the notation $u_{i_1 i_2}$ to represent the approximate value to $u(i_1h_1, i_2h_2)$.

3. Discretisation of the differential problem

Let B be the operator acting on $u_{i_1 i_2}$ and defined by one of the four expressions (4) below

$$B_i \equiv \begin{cases} \equiv \frac{1}{h_i^2}\delta_{x_i}(\alpha_i(x_i)\delta_{x_i}) - c_i(x_i) & \text{(4a)} \\ \equiv \frac{1}{h_i^2}\delta_{x_i}(\alpha_i(x_i)\delta_{x_i}) - \frac{1}{h_i^2}\alpha_i\left(x_i + (1-1)^{j+1}\frac{h_i}{2}\right)E_{x_i}^{(-1)^{j+1}} - c_i(x_i) & \text{(4b)} \\ \equiv \frac{2\alpha_i\left(x_i + \frac{h_i}{2}\right)}{h_i^2}(\Delta_{x_i} - h_i\sigma_i^0) - c_i(x_i) & \text{(4c)} \\ \equiv -\frac{2\alpha_i\left(x_i - \frac{h_i}{2}\right)}{h_i^2}(\nabla_{x_i} + h_i\sigma_i^1) - c_i(x_i) & \text{(4d)} \end{cases}$$

In the expressions above δ_{x_i}, E_{x_i}, Δ_{x_i} and ∇_{x_i} are the central difference, the shifting, the forward difference and the backward difference operators in x_i-direction respectively. The expression to be used for B_i in a specific case depends on the value of i, the position of the point $x \equiv (x_1, x_2) \equiv \equiv (i_1h_1, i_2h_2)$ in $\bar{R}$ and the problem being solved. In each case arising the appropriate expression for the operator B_i is used according to Table I

TABLE I

Expression	Problem	i	Ranges or values for i_1	Ranges or values for i_2
4a	PI	1	$[2, N_1 - 1]$	$[1, N_2 - 1]$
		2	$[1, N_1]$	$[2, N_2 - 2]$
	PII	1	$[2, N_1 - 1]$	$[0, N_2 - 1]$
		2	$[1, N_1]$	$[1, N_2 - 2]$
	PIII	1	$[2, N_1 - 2]$	$[0, N_2]$
		2	$[1, N_1 - 1]$	$[1, N_2 - 1]$
	PIV	1	$[2, N_1 - 1]$	$[0, N_2]$
		2	$[1, N_1]$	$[1, N_2 - 1]$
4b	PI	1	1	$[1, N_2 - 1]$
		2	$[1, N_1]$	$1, N_2 - 1$
	PII	1	1	$[0, N_2 - 1]$
		2	$[1, N_1]$	$N_2 - 1$
	PIII	1	$1, N_1 - 1$	$[0, N_2]$
	PIV	1	1	$[0, N_2]$
4c	PII	2	$[1, N_1]$	0
	PIII	2	$[1, N_1 - 1]$	0
	PIV	2	$[1, N_1]$	0
4d	PI	1	N_1	$[1, N_2 - 1]$
	PII	1	N_1	$[0, N_2 - 1]$
	PIII	2	$[1, N_1 - 1]$	N_2
	PIV	1	N_1	$[0, N_2]$
		2	$[1, N_1]$	N_2

Using the appropriate forms of the operator B_i given by expressions (4) the differential problem (1) — (3) is replaced by the following discrete one

$$(5) \qquad (B_1 + B_2)u_{i_1 i_2} = \varphi_{i_1 i_2}, \quad x \equiv (x_1, x_2) \equiv (i_1 h_1, i_2 h_2) \in \bar{R} - \partial R_D$$

where the values for $\varphi_{i_1 i_2}$ are easily obtained from the R.H.S. of equation (1) and the corresponding boundary conditions. It can be found out that equations (5) have a truncation error $0(h_1^2) + 0(h_2^2)$ except in the case where the point x involved lies on ∂R_N. Then according to which side of the rectangle the point x lies on, the corresponding exponent in h_i is decreased by one.

If we define the integers K_i and $L_i \mid i \in I$ according to Table II

TABLE II

Problem	K_1	K_2	L_1	L_2
P I	1	1	N_1	$N_2 - 1$
P II	1	0	N_1	$N_2 - 1$
P III	1	0	$N_1 - 1$	N_2
P IV	1	0	N_1	N_2

and denote the expression $L_i - K_i + 1$ by $M_i | i \in I$ we can readily see that the totality of equations (5) can be written in matrix from as follows

$$(6) \qquad (A_1^* + A_2^*)\, u^* = \Phi^*$$

In matrix equation (6) above A_i^* are known matrices of order $M_1 M_2$, u^* is an unknown $M_1 M_2$-dimensional vector of the form

$$u^* = (u_{K_1 K_2}, u_{K_1+1, K_2}, \ldots, u_{L_1 K_2}, u_{K_1, K_2+1}, u_{K_1+1, K_2+1}, \ldots, u_{L_1 L_2})^T$$

and Φ^* a known $M_1 M_2$-dimensional vector given by

$$\Phi^* = -h_1 h_2 (\varphi_{K_1 K_2}, \varphi_{K_1+1, K_2}, \ldots, \varphi_{L_1 K_2}, \varphi_{K_1, K_2+1}, \varphi_{K_1+1, K_2+1}, \ldots, \varphi_{L_1 L_2})^T.$$

The matrices A_i^* have the following product forms

$$A_1^* = J_2 \otimes H_1^* \text{ and } A_2^* = H_2^* \otimes J_1$$

with J_i being unit matrix of order M_i, H_i^* a matrix of order M_i being given below and the symbol $\otimes$ denoting tensor product as is defined in

HALMOS [6]. The matrices H_i^* are given for the four different problems as follows

$$(7)\quad H_1^* = \frac{h_2}{h_1}\begin{bmatrix} \alpha_1^{1/2}+\alpha_1^{3/2}+c_1^1h_1^2 & -\alpha_1^{3/2} & & \\ -\alpha_1^{3/2} & \alpha_1^{3/2}+\alpha_1^{5/2}+c_1^2h_1^2 & -\alpha_1^{5/2} & \\ & \ddots & \ddots & -\alpha_1^{M_1-1/2} \\ & & -2\alpha_1^{M_1-1/2} & 2\alpha_1^{M_1-1/2}(1+h_1\sigma_1^1)+c_1^{M_1}h_1^2 \end{bmatrix}$$

for problems PI, PII and PIV,

$$(8)\quad H_i^* = \frac{h_k}{h_i}\begin{bmatrix} \alpha_1^{1/2}+\alpha_i^{3/2}+c_i^1h_i^2 & -\alpha_1^{3/2} & & \\ -\alpha_i^{3/2} & \alpha_i^{3/2}+\alpha_i^{5/2}+c_i^2h_i^2 & -\alpha_i^{5/2} & \\ & \ddots & \ddots & -\alpha_i^{M_i-1/2} \\ & & -\alpha_i^{M_i-1/2} & \alpha_i^{M_i-1/2}+\alpha_i^{M_i-1/2}+c_i^{M_i}h_i^2 \end{bmatrix}$$

for $k \in I - \{i\}$ and problem P III for $i = 1$ and problem P I for $i = 2$,

$$(9)\quad H_2^* = \frac{h_1}{h_2}\begin{bmatrix} 2\alpha_2^{1/2}(1+h_2\sigma_2^0)+c_2^0h_2^2 & -2\alpha_2^{1/2} & & \\ -\alpha_2^{1/2} & \alpha_2^{1/2}+\alpha_2^{3/2}+c_2^1h_2^2 & -\alpha_2^{3/2} & \\ & \ddots & \ddots & -\alpha_2^{M_2-3/2} \\ & & -\alpha_2^{M_2-3/2} & \alpha_2^{M_2-3/2}+\alpha_2^{M_2-1/2}+c_2^{M_2-1}h_2^2 \end{bmatrix}$$

for problem P II and finally

$$(10)\quad H_2^* = \frac{h_1}{h_2}\begin{bmatrix} 2\alpha_2^{1/2}(1+h_2\sigma_2^0)+c_2^0h_2^2 & -2\alpha_2^{1/2} & & \\ -\alpha_2^{1/2} & \alpha_2^{1/2}+\alpha_2^{3/2}+c_2^1h_2^2 & -\alpha_2^{3/2} & \\ & \ddots & \ddots & -\alpha_2^{M_2-3/2} \\ & & -2\alpha_2^{M_2-3/2} & 2\alpha_2^{M_2-3/2}(1+h_2\sigma_2^1)+c_2^{M_2-1}h_2^2 \end{bmatrix}$$

for problems P III and P IV. In all matrices above α_l^i and c_i^m stand for the values of $\alpha_i\,(lh_i)$ and $c_i(mh_i)$ respectively for all possible values of l and m.

4. Transformation of the linear system

Referring to the matrices H_i^* we define the matrices C_i of order M_i as follows

$$C_i = \begin{bmatrix} a & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & b \end{bmatrix}, \quad i = 1,2$$

with the meaning of a and b given in Table III

TABLE III

i		$P\ I$	$P\ II$	$P\ III$	$P\ IV$
1	a	1	1	1	1
	b	$\sqrt{2}$	$\sqrt{2}$	1	$\sqrt{2}$
2	a	1	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
	b	1	1	$\sqrt{2}$	$\sqrt{2}$

Together with the matrices C_i defined previously we also define the matrix C by the relationship

$$C = C_2 \otimes C_1$$

and multiply equation (6) from left by C^{-1}. Thus we obtain

$$(11)\qquad (A_1 + A_2)\, u = \Phi$$

where

$$A_i = C^{-1}A_i^*C, \quad u = C^{-1}u^* \text{ and } \Phi = C^{-1}\Phi^*$$

It can be proved that

$$A_1 = J_2 \otimes H_1,\ A_2 = H_2 \otimes J_1$$

where the matrices H_i are given by the relationships

$$H_i = C_i^{-1}H_i^*C_i$$

and are symmetric. Using all the previous results concerning the new matrices A_i and H_i introduced above it can be proved that the matrices A_i possess the following properties

i) they are symmetric

ii) they commute and

iii) the eigenvalues of A_i are those of H_i (or equivalently those of H_i^*) with each eigenvalue repeated M_k times ($k \in I - \{i\}$).

For the numerical solution of linear system (11) by using E.A.D.I. methods we need (see e,g, [1] and [5] as strict as possible non negative bounds for the eigenvalues of the matrices A or, which is the same, for the eigenvalues of the matrices H_i or H_i^*. These bounds are given in the next section

5. Bounds for the eigenvalues of the matrices H_i^*

Let

$$\underline{\alpha}_i = \min \alpha_i^1, \quad \underline{c}_i = \min c_i''' \quad \text{and}$$
$$\bar{\alpha}_i = \max \alpha_i^1, \quad \bar{c}_i = \max c_i'''$$

be the extreme values, for all permissible values, of α_i^1 and c_i''' in each case. Let also λ be any eigenvalue of the matrix H_i^*. In a way analogous to that developed in [3] it can be shown that λ is bounded as follows

$$\underline{\alpha}_i \frac{h_k}{h_i} \lambda_{\min} + \underline{c}_i h_1 h_2 \leqq \lambda \leqq \bar{\alpha}_i \frac{h_k}{h_i} \lambda_{\max} + \bar{c}_i h_1 h_2 \tag{12}$$

where $k \in I - \{i\}$ and $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and the maximum eigenvalues of a matrix A or strict lower and upper bounds for these eigenvalues respectively. The matrix A which differs from problem to problem is defined in the analysis which follows.

For the eigenvalues λ of the matrix H_i^* given by (7) we have from (12) that

$$\underline{\alpha}_1 \frac{h_2}{h_1} \lambda_{\min} + \underline{c}_1 h_1 h_2 \leqq \lambda \leqq \bar{\alpha}_1 \frac{h_2}{h_1} \lambda_{\max} + \bar{c}_1 h_1 h_2 \tag{13}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are strict lower and upper bounds for the eigenvalues of the matrix A defined by

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 + 2h_1\sigma_1^1 \end{bmatrix} \tag{14}$$

To find $\lambda_{\min}$ and $\lambda_{\max}$ we work as follows. The eigenvalues of the matrix A are the roots of the determinental equation

$$g(\lambda) = \det\,(A - \lambda I) = (2 + 2h_1\sigma_1^1 - \lambda) T_{M_1 - 1}(\lambda) - 2T_{M_1-2}(\lambda) = 0$$

where $T_M(\lambda)$ is the determinant of order M given by

$$T_M(\lambda) = \begin{vmatrix} 2-\lambda & -1 & & & \\ -1 & 2-\lambda & -1 & & \\ & & \ddots & & \\ & & -1 & 2-\lambda & -1 \\ & & & -1 & 2-\lambda \end{vmatrix}$$

From this determinant we can readily obtain that

$$T_M(\lambda) = (2 - \lambda) T_{M-1}(\lambda) - T_{M-2}(\lambda)$$

with $T_0(\lambda) = 1$ and $T_1(\lambda) = 2 - \lambda$ which, in turn, give that $T_M(0) = M + 1$. Therefore $g(0) = 2(h_1 M_1 \sigma_1^1 + 1) > 0$. Thus if we put

$$\lambda = 4\sin^2\frac{\varphi}{2}, \quad \varphi \in (0, \pi) \tag{15}$$

we can prove that

$$T_M(\lambda) = \frac{\sin\,(M+1)\varphi}{\sin\,\varphi}$$

and also that

$$g(\lambda) = \frac{2}{\sin\,\varphi} g_1(\varphi)$$

where

$$g_1(\varphi) = h_1\sigma_1^1 \sin\, M_1\,\varphi + \cos M_1\varphi \sin\,\varphi$$

We distinguish two cases according to whether σ_1^1 is zero or not

1st case: $\sigma_1^1 = 0$

In this case $g_1(\varphi) = 0$ implies that $\cos\, M_1\, \varphi = 0$ which gives

$$\varphi = (4k+1)\pi/2M_1 | k = 0(1)\left[\frac{M_1 - 1}{2}\right] \quad \text{and}$$
$$\varphi = (4k-1)\pi/2M_1 | k = 1(1)\left[\frac{M_1}{2}\right] \tag{16}$$

We observe that the number of the eigenvalues of H_1^* which are obtained by substituting expressions (16) into (15) is $\left[\frac{M_1-1}{2}\right]+1+\left[\frac{M_1}{2}\right]=M_1$ and this is equal to the total number of the eigenvalues of H_1^*. Therefore

$$\lambda_{min}=4\sin^2\frac{\pi}{4M_1} \quad\text{and}\quad \lambda_{max}=4\cos^2\frac{\pi}{4M_1}$$

These values for λ_{min} and λ_{max} are used apparently in relationships (13)

2^{nd} sase: $\sigma_1^1>0$

In this case we can get

$$g_1\left(\frac{k\pi}{M_1}\right)=(-1)^k\sin\frac{k\pi}{M_1}\ |k=1(1)M_1-1$$

which implies that an odd number of roods of $g_1(\varphi)$ lies in each interval $\left(\frac{k\pi}{M_1},\frac{(k+1)\pi}{M_1}\right)|k=1(1)M_1-2$. In addition to that we observe that if we differentiate $g_1(\varphi)$ with respect to φ we obtain

$$g_1'(\varphi)=(h_1M_1\sigma_1^1+\cos\varphi)\cos M_1\varphi-M_1\sin M_1\varphi\sin\varphi$$

Thus we have that $g_1(0)=0$, $g_1'(0)=h_1M_1\sigma_1^1+1>0$ and $g_1\left(\frac{\pi}{M_1}\right)=-\left(h_1M_1\sigma_1^1+\cos\frac{\pi}{M_1}\right)<0.$

These last relationships imply that an odd number of roods lies in the interval $\left(0,\frac{\pi}{M_1}\right)$. Moreover if we put

$$\varphi=\pi+i\theta,\quad \theta>0 \text{ and } i=\sqrt{-1}$$

we obtain

$$g_1(\varphi)=(-1)^{M_1}ig_2(\theta)$$

where

$$g_2(\theta)=h_1\sigma_1^1\,\mathrm{sh}M_1\theta-\mathrm{ch}\,M_1\theta\,\mathrm{sh}\,\theta$$

Thus we have that to every positive root of $g_2(\theta)$ there corresponds a root of $g_1(\varphi)$ greater than 4. Because of the relationships $g_2(0)=0$ and $\lim_{\theta\to+\infty} g_2(\theta)=+\infty$ which are readuly obtained we come to the conclusion that if we choose $\theta_0=\mathrm{arcsh}\,(h_1\sigma_1^1)$ we get $g_2(\theta_0)<0$. Consequently an odd number of roots of $g_2(\theta)$ is positive. Thus we have succeded in determining the position of all the roots of $g_1(\varphi)$. Therefore in this present case we can take $\varphi=\frac{\pi}{2M_1}$, because of the relationship $g_1\left(\frac{\pi}{2M_1}\right)=h_1\sigma_1^1>0$ which is valid. Hence

$$\lambda_{min}=4\sin^2\frac{\pi}{4M_1}$$

For λ_{max} we can take

$$\lambda_{max}=\min\{||A||_1,\ ||A||_\infty\}=\begin{cases}=4+2h_1\sigma_1^1 & \text{if } \sigma_1^1<\frac{1}{2h_1}\\ =5 & \text{if } \frac{1}{2h_1}\le\sigma_1^1\le\frac{1}{h_1}\\ =3+2h_1\sigma_1^1 & \text{if } \frac{1}{h_1}<\sigma_1^1\end{cases}$$

For the matrix H_i^* given by (8) we have that its eigenvalues λ are bounded as in relationships (12) with λ_{min} and λ_{max} being referred to the eigenvalues of the matrix A given by

$$A=\begin{bmatrix}2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1\\ & & & -1 & 2\end{bmatrix}$$

As is known the eigenvalues of A above are given by the expressions $4\sin^2\frac{k\pi}{2(M_i+1)}\ |k=1(1)M_i$ therefore

$$\lambda_{min}=4\sin^2\frac{\pi}{2(M_i+1)} \quad\text{and}\quad \lambda_{max}=4\cos^2\frac{\pi}{2(M_i+1)}$$

For the matrix H_2^* given by (9) we have that its eigenvalues are bounded as follows

(17) $$\underline{\alpha}_2\frac{h_1}{h_1}\lambda_{min}+\underline{c}_2h_1h_2\le\lambda\le\bar{\alpha}_2\frac{h_1}{h_2}\lambda_{max}+\bar{c}_2h_1h_2$$

with λ_{min} and λ_{max} being referred to the eigenvalues of the matrix A given by

$$A=\begin{bmatrix}2+2h_2\sigma_2^0 & -2 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1\\ & & & -1 & 2\end{bmatrix}$$

This present case is obviously analogues to the one where A was given by (14).

Therefore we can obtain similar results. More specifically if $\sigma_0^2=0$

$$\lambda_{min}=4\sin^2\frac{\pi}{2M_2} \quad\text{and}\quad \lambda_{max}=4\cos^2\frac{\pi}{2M_2}$$

while if $\sigma_2^0 > 0$

$$\lambda_{\min}=4\sin\frac{\pi}{2M_2} \text{ and } \lambda_{\max}=\begin{cases}= 4+2h_2\sigma_2^0 & \text{if } \sigma_2^0<\frac{1}{2h_2}\\ = 5 & \text{if } \frac{1}{2h^2}\leqq\sigma_2^0\leqq\frac{1}{h_2}\\ = 3+2h_2\sigma_2^0 & \text{if } \frac{1}{h_2}<\sigma_2^0\end{cases}$$

Finally, for the matrix H_2^* given by (10) we have that its eigenvalues λ are bounded as in (17) where now $\lambda_{\min}$ and $\lambda_{\max}$ are referred to the eigenvalues of the matrix A given by

$$A=\begin{bmatrix}2+2h_2\sigma_6^0 & -2 & & & \\ -1 & 2 & -1 & & \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ & -1 & 2 & -1 \\ & & -2 & 2+h_0\sigma_9^3\end{bmatrix}$$

This problem, however, was solved in [1] and was also presented in [2] where strict bounds for the eigenvalues of the matrix A above, that is the numbers $\lambda_{\min}$ and $\lambda_{\max}$, were found. Therefore here we simply quote the results obtained in [1].

So if we put $\Sigma=\sigma_2^0+\sigma_2^1$ and $\sigma=\max\{\sigma_2^0,\ \sigma_2^1\}$ we have that

$$\lambda_{\min}=4\sin^2\frac{\varphi_0}{2}$$

where

$$\varphi_0=\begin{cases}=\frac{\pi}{2(M_2-1)} & \text{if } \sin^2\frac{\pi}{2(M_2-1)}\leqslant h_2^2\sigma_2^0\sigma_2^1\\ =\operatorname{Arcsin}\left(h_2\sqrt{\sigma_2^0\sigma_2^1}\right) & \text{if } 0<h_2^2\sigma_2^0\sigma_2^1<\sin^2\frac{\pi}{2(M_2-1)}\\ =\frac{1}{(M_2-1)}\operatorname{Arctn}\left(h_2\Sigma/\sin\left(\frac{\pi}{2(M_2-1)}\right)\right) & \text{if } \sigma_2^0\sigma_2^1=0\end{cases}$$

and

$$\lambda_{\max}=\begin{cases}=4+2h_2\sigma & \text{if } \sigma<\frac{1}{2h_2}\\ =5 & \text{if } \frac{1}{2h_2}\leqslant\sigma\leqslant\frac{1}{h_2}\\ =3+2h_2\sigma & \text{if } \frac{1}{h_2}<\sigma\end{cases}$$

6. Numerical example

We consider the differential equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}=0,\quad x\in R \tag{18}$$

where R is the interior of the rectangle with vertices (0, 0), (1.2, 0), (1.2,1.0) and (0,1.0). The solution u of the above equation is assumed to satisfy the following boundary conditions

$$\begin{aligned} u&=0, & x&\in\partial R_D=\partial R_1^0\cup\partial R_1^1\\ \frac{\partial u}{\partial x_2}&=0, & x&\in\partial R_N\in=\partial R-\partial R_D\end{aligned}\tag{19}$$

According to the notation in section 2 we obviously have that

$$\alpha_1(x_1)\equiv\alpha_2(x_2)\equiv 1,\ c_1(x_2)\equiv c_2(x_2)\equiv 0,\quad \sigma_2^0=\sigma_2^1=0$$

Consequently it is apparent that our numerical example is a P III type problem.

To solve differential problem (18)—(19) numericaly we impose a uniform grid of mesh sizes $h_1=l_1/N_1=1.2/12=0.1$ and $h_2=l_2/N_2=1.0/10=$ $=0.1$ in the corresponding arbitrarily. If $u_{i_1 i_2}\equiv u(i_1h_1,\ i_2h_2)|i_1=(1(1)11,$ $i_2=0(1)10$ are the numerical solutions at the nodes $(i_1h_1,\ i_2h_2)$ of the grid then the discrete problem we have to solve is the following

$$(A_1^*+A_2^*)u^*=0 \tag{20}$$

In matrix equation (20) we have that

$$A_1^*=J_2\otimes H_1^*,\ A_2^*=H_2^*\otimes J_1$$

with $J_i|i=1{,}2$ being the unit matrices of order 11, $H_i^*|i=1{,}2$ are 11×11 matrices of the forms

$$H_1^*=\begin{bmatrix}2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ & -1 & 2 & -1 \\ & & -1 & 2\end{bmatrix},\quad H_2^*=\begin{bmatrix}2 & -2 & & & \\ -1 & 2 & -1 & & \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ & -1 & 2 & -1 \\ & & -2 & 2\end{bmatrix}$$

and u^* the unknown vector of the numerical solutions at the nodes of the grid of the form

$$u^*=(u_{1,0},\ u_{2,0},\ \ldots\ u_{11,0},\ u_{1,1},\ \ldots,\ u_{11,10})^T$$

The great advantage of the numerical example considered is that both the differential problem (18)—(19) and its corresponding discret one (20) have apparently as their solutions the zero solution. This implies that the m^{th} vector iteration approximation to u^* coming from the application of the E.A.D.I. method will coincide with the error vector $\varepsilon^{*(m)}$ of the same iteration.

In order to be able to solve numerically matrix equation (20) by using E.A.D.I. methods we have to transform it according to the theory developed in section 4. Thus we multiply equation (20) from the left by $C^{-1} = C_2^{-1} \otimes C_1^{-1}$ where $C_i^* | i = 1{,}2$ are diagonal matrices of order 11 defined by

$$C_1 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad C_2^* = \begin{bmatrix} \sqrt{2} & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 1 & \\ & & & & & \sqrt{2} \end{bmatrix}$$

Therefore equation (20) is transformed into

$$(A_1 + A_2)u = 0 \tag{21}$$

where

$$A_i = C^{-1}A_i^*C | i = 1{,}2, \quad u = C^{-1}u^*$$

It is readily seen that

$$A_1 = J_2 \otimes H_1, \quad A_2 = H_2 \otimes J_1$$

where

$$H_1 = H_1^*, \; H_2 = C_2^{-1}H_2^*C_2 = \begin{bmatrix} 2 & -\sqrt{2} & & & & \\ -\sqrt{2} & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -\sqrt{2} \\ & & & & -\sqrt{2} & 2 \end{bmatrix}$$

So the smallest and the largest eigenvalues of H_1 and H_2, and consequently of A_1 and D_2, are the following

$$\lambda_{\min_1} = 4 \sin \frac{2\pi}{24}, \quad \lambda_{\max_1} = 4 \cos \frac{2\pi}{24}$$

$$\lambda_{\min_2} = 0 \quad , \quad \lambda_{\max_2} = 4$$

The E.A.D.I. scheme corresponds now to matrix equation (21) is the following (see [5])

$$\begin{aligned} &(I + r_{m+1}A_1)u^{(m+1/2)} = [(I + r_{m+1}A_1) - \omega r_{m+1}(A_1 + A_2)]u^{(m)} \\ &\hspace{20em} | m = 0, 1, 2, \ldots \\ &(I + r_{m+1}A_2)u^{(m+1)} = u^{(m+1/2)} + r_{m+1}A_2u^{(m)} \end{aligned} \tag{22}$$

In scheme (22) I is the unit matrix of order 121, $u^{(m)}$ is the m^{th} iteration approximation to the solution u of (21), with $i^{(0)}$ arbitrary, ω is the extrapolation parameter, $r_{m+1} = r_n \; | n = 1(1)n_0 \;\; (n = m + 1 - n_0[m/n_0])$ are n_0 positive iteration parameters and $u^{(m+1/2)}$ an intermediate approximation to $u^{(m+1)}$. The optimum parameters to be used in connection with scheme (22) can be found in the way described in [5]. Thus by using the set of Samarskii Andreyev parameters we can finally obtain the following optimum results

$$\begin{aligned} &\omega = 1.6449290, \quad n_0 = 2, \quad \rho = 0.4492767 \\ &r_1 = 7.9238896, \quad r_2 = 0.5776515 \end{aligned} \tag{23}$$

with ρ being the optimum amplification factor of the procedure.

Denoting by $u^{*(m)} \; | m = 0, 1, 2, \ldots$ the sequence of approximate vectors to the exact solution u^* of equation (20) which are related to the coorresponding $u^{(m)}$ through the relationship

$$u^{*(m)} = Cu^{(m)}$$

we can easily find out the corresponding error vectors $\varepsilon^{*(m)}$ and $\varepsilon^{(m)}$ will satisfy the same relationship namely

$$\varepsilon^{*(m)} = C\varepsilon^{(m)} \tag{24}$$

This is because the exact solutions of both equations (20) and (21) are such that $u^* = u = 0$. Thus we come to the conclusion that in order to reduce the second norm of the initial error vector $\varepsilon^{*(0)}$ by a factor e we have to perform a number of s cycles with n_0 iterations of type (22) within each cycle. The number s is found as follows. Since we want to have $\|\varepsilon^{*(s \cdot n_0)}\| / \|\varepsilon^{*(0)}\| \leq e$ and by virtue of relationship (24) the following are valid

$$\varepsilon^{(s \cdot n_0)} = \left(\prod_{n=1}^{n_0} T_n\right)^s \varepsilon^{(0)}$$

where T_i is the iteration matrix at the i iteration of scheme (22)

$$C^{-1}\varepsilon^{*(s\cdot n_0)} = \left(\prod_{n=1}^{n_0} T_n\right)^s C^{-1}\varepsilon^{*(0)},$$

$$\varepsilon^{*(s\cdot n_0)} = C\left(\prod_{n=1}^{n_0} T_n\right)^s C^{-1}\varepsilon^{*(0)},$$

$$\|\varepsilon^{*(s\cdot n_0)}\| \leqslant \|C\| \cdot \left\|\prod_{n=1}^{n_0} T_n\right\|^s \cdot \|C^{-1}\| \cdot \|\varepsilon^{*(0)}\|$$

Since

$$\|C\| = \sqrt{2}, \quad \|C^{-1}\| = 1 \text{ and } \left\|\prod_{n=1}^{n_0} T_n\right\| \leqslant \rho,$$

then

$$\|\varepsilon^{*(s\cdot n_0)}\| \leqslant \sqrt{2}\rho^s \|\varepsilon^{*(0)}\|$$

or

$$\|\varepsilon^{*(s\cdot n_0)}\| / \|\varepsilon^{*(0)}\| \leqslant \sqrt{2}\rho^s.$$

Whence it is sufficient to require

$$\rho^s \leqslant e/\sqrt{2}$$

that is

$$s = [\ln(e/\sqrt{2})/\ln(\rho)] + 1$$

Taking e.g. $e = 10^{-6}$ we can fiind that $s = 18$.

This special example was run on the UNIVAC 1106 Computer of the University of Salonika by choosing as $u^{*(0)}$ the vector with all its components equal to unity. Thus we found out the reduction by a factor of 10^{-6} in the norm of the initial error vector was achieved after 15 cycles which was far better than the number of 18 cycles we had expected from the theory.

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