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ON INTERPOLATION OPERATORS (I)

(A proof of Jackson's theorem for differentiable functions)

by

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1. Introduction. In the recent years many authors devoted themselves to giving new proofs of the famous theorem of JACKSON [2], TIMAN [5] and TELYAKOVSKI [4] GOPENGAUZ [1] on the approximation of continuous function defined on $[-1, 1]$ by algebraic polynomials. A number of constructive proofs now occur in the literature. Many of them involve the construction of procedures which interpolate the function to be approximated at a given set of nodes. Our task in this series is to give interpolation operators which provide with the new proofs of the above theorems for differentiable functions. In this first communication we construct the operators $V_{nr}(f, x)$ $(r = 0,1)$ and study their approximating properties towards functions $f(x)$ given on $[-1, 1]$. We find that $V_{nr}(f, x)$ satisfies Jackson's inequality when $f(x) \in C^1[-1, 1]$[1]. The operators $V_{n0}(f, x)$ satisfy Timan's inequality for functions $f(x) \in C[-1, 1]$. At the same time the Trigub's inequality on the derivative of approximating polynomials is also satisfied. We mention that $V_{n1}(f, x)$ does not appear to be strong enough so as to satisfy Timan's inequality. However, a little modification may sharpen the estimates, but that does not fulfil our aim of proving Timan's theorem for differentiable functions. For this we have to consider other operators and this will be the content of our next communication. The source of these investigations is the work of KIS-VERTESI [3] which all depends on the identity of A. H. TURECKII [7]. We shall also obtain few identities which, apart from their use here, are interesting in their own right.

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[1] $C^1[-1, 1]$ denotes the class of functions whose first derivatives are continuous on $[-1, 1]$.

2. The operator $V_{nr}(f, x)$. Let $-1 \leq x \leq 1$, $\cos t = x$ and $\cos t_{kn} = x_{kn}$ with

$$t_{kn} = \frac{2k\pi}{2n+1}, \quad k = \overline{0, n}\ ^{2)}; \qquad n = 1, 2, \ldots \tag{2.1}$$

Further for $k = \overline{-n, n}$, let

$$l_{kn}(t) = \frac{\sin \frac{2n+1}{2}(t - t_{kn})}{(2n+1)\sin \frac{1}{2}(t - t_{kn})} \tag{2.2}$$

and

$$u_{kn}(t) = \frac{1}{3}\,[25\, l_{kn}^4(t) - 32\, l_{kn}^5(t) + 10\, l_{kn}^6(t)]. \tag{2.3}$$

Then for any arbitrary function $f(x)$ given on $[-1, 1]$, we define the operators

$$V_{nr}(f, x) = \sum_{\nu=0}^{r} (x+1)^{\nu} f^{(\nu)}(-1) + \sum_{k=0}^{n} \left[\sum_{\nu=0}^{r} (x - x_{kn})^{\nu} f^{(\nu)}(x_{kn}) - (x+1)^{\nu} f^{(\nu)}(-1) \right] v_{kn}(x), \quad r = 0, 1, \tag{2.4}$$

where

$$v_{on}(x) = u_{on}(t)\,; \quad v_{kn}(x) = u_{kn}(t) + u_{-kn}(t), \qquad k = \overline{1, n}. \tag{2.5}$$

The properties of the operators $V_{nr}(f, x)$ are given in

THEOREM 1. For $r = 0, 1$

(a) *$V_{nr}(f, x)$ is an algebraic polynomial of degree $\leq 6n + 1$ in x,*

(b) $V_{nr}^{(\nu)}(f, x_{kn}) = f^{(\nu)}(x_{kn})$ $\quad (\nu = \overline{0, r})$, $\quad k = \overline{0, n}$.

Proof. Since

$$\frac{\sin \frac{2n+1}{2}(t - t_{kn})}{\sin \frac{1}{2}(t - t_{kn})} = 1 + 2\sum_{j=1}^{n} \cos j(t - t_{kn}), \tag{2.6}$$

therefore

$$u_{kn}(t) = \sum_{j=0}^{6n} C_j \cos j(t - t_{kn}), \qquad k = \overline{-n, n},$$

[2)] $k = \overline{0, n}$ stands for $k = 0, 1, 2, \ldots, n$.

where C_j are some numbers. Hence

$$v_{on}(x) = \sum_{j=0}^{6n} C_j \cos jt \ \text{ and } \ v_{kn}(x) = 2\sum_{j=0}^{6n} C_j \cos jt \cos jt_{kn}, \qquad k = \overline{1, n},$$

from which the assertion (a) follows.

To prove (b) it is sufficient to show that

$$v_{kn}(x_{jn}) = \delta_{kj} \ \text{ and } \ v'_{kn}(x_{jn}) = 0 \qquad (k, j = \overline{0, n}). \tag{2.7}$$

We observe that

$$l_{kn}(t_{jn}) = \delta_{kj} \quad (k, j = 0, \pm 1, \pm 2, \ldots, \pm n) \tag{2.8}$$

and so on account of (2.3)

$$u_{kn}(t_{jn}) = \delta_{kj} \quad (j, k = 0, \pm 1, \pm 2, \ldots, \pm n) \tag{2.9}$$

from which the first part of (2.7) follows after using (2.5). Further from (2.3) on differentiation

$$u'_{kn}(t) = \frac{1}{3}\,[100\, l_{kn}^3(t) - 160\, l_{kn}^4(t) + 60\, l_{kn}^5(t)]\, l'_{kn}(t),$$

$$u''_{kn}(t) = \frac{1}{3}\,[100\, l_{kn}^3(t) - 160\, l_{kn}^4(t) + 60\, l_{kn}^5(t)]\, l''_{kn}(t) + \frac{1}{3}\,[300\, l_{kn}^2(t) - 640\, l_{kn}^3(t) + 300\, l_{kn}^4(t)]\, l'^2_{kn}(t),$$

from which on using $l_{kn}(t_{kn}) = 0$, which is an easy consequence of (2.6), it follows that

$$u'_{kn}(t_{jn}) = u''_{kn}(t_{jn}) = 0 \qquad (k, j = 0, \pm 1, 2, \pm 3, \ldots, \pm n) \tag{2.10}$$

Now differentiating twice the formula $v_{on}(\cos t) = u_{on}(t)$ we have

$$-\sin t \ \ v'_{on}(\cos t) = u'_{on}(t), \tag{2.11}$$

$$\sin^2 t \ \ v''_{on}(\cos t) - \cos t \ \ v'_{on}(\cos t) = u''_{on}(t). \tag{2.12}$$

Using (2.10), we have from (2.11)

$$v'_{on}(x_{jn}) = 0, \qquad j = \overline{1, n} \tag{2.13}$$

and from (2.12)

$$v'_{on}(x_{on}) = 0, \ v''_{on}(x_{jn}) = 0, \qquad j = \overline{1, n}. \tag{2.14}$$

Similarly if we differentiate twice the formula

$$v_{kn}(\cos t) = u_{kn}(t) + u_{-kn}(t), \qquad k = \overline{1, n},$$

and use (2.10), then we get

$$v'_{kn}(x_{jn}) = 0, \quad j = \overline{0,\ n} \text{ and} \tag{2.15}$$

$$v''_{kn}(x_{jn}) = 0, \quad j = 1,\ n. \tag{2.16}$$

Hence combining (2.13), (2.14), (2.15) and (2.16), we get

$$\begin{cases} v'_{kn}(x_{jn}) = 0, & (k,\ j = \overline{0,\ n}) \\ v''_{kn}(x_{jn}) = 0 & (k = \overline{0,\ n}) \\ & (j = \overline{1,\ n}) \end{cases} \tag{2.17}$$

In the same way it follows that

$$v'''_{kn}(x_{jn}) = 0, \quad (k = \overline{0,\ n}, \quad j = \overline{1,\ n}).$$

3. Some identities. We have the following

Lemma 1. *Let*

$$S_m = \sum_{k=-n}^{n} l^m_{kn}(t),$$

then

$$S_3 = \frac{1}{(2n+1)^2}[(3n^2 + 3n + 1) + n(n+1)\cos(2n+1)t], \tag{3.1}$$

$$S_4 = \frac{1}{3(2n+1)^3}[(8n^2 + 8n + 3) + 4n(n+1)\cos(2n+1)t], \tag{3.2}$$

$$S_5 = \frac{1}{4!\,(2n+1)^4}[(230n^4 + 460n^3 + 370n^2 + 140n + 24) + 2(76n^4 + 152n^3 + 104n^2 + 28n)\cos(2n+1)t + 2n(n^2-1)(n+2)\cos(4n+2)t], \tag{3.3}$$

$$S_6 = \frac{1}{5!\,(2n+1)^4}[12(88n^4 + 176n^3 + 142n^2 + 54n + 10) + (416n^4 + 832n^3 + 584n^2 + 168n)\cos(2n+1)t + (16n^4 + 32n^3 + 4n^2 - 12n)\cos(4n+2)t]. \tag{3.4}$$

For the sums S_3 and S_4, there holds the identity

$$4S_3 - 3S_4 = 1 \tag{3.5}$$

For the sums of higher orders we have the identity

(3.6)

$$\frac{1}{3}[25S_4 - 32S_5 + 10S_6] = 1 + \frac{10n(n+1)}{9(2n+1)^4}[3 - 4\cos(2n+1)t + \cos(4n+2)t].$$

The identity (3.5) was first given by A. H. TURECKI [7]. We calculate S_5 and S_6. Following KIS-VERTESI [3], we have, for a positive integer m

$$(2n+1)^{m-1}\sum_{k=-n}^{n} l^m_{kn}(t) = C_{0,m} + 2\sum_{j=1}^{\left[\frac{mn}{2n+1}\right]} C_{(2n+1)j,\,m}\cos(2n+1)jt \tag{3.7}$$

where the numbers $C_{j,m}$ are such that $C_{j,m} = C_{-j,m}$, $j = \overline{1,\ mn}$ and satisfy

$$\sum_{j=-mn}^{mn} C_{j,m}Z^j = Z^{-mn}(1 - Z^{2n+1})^m \sum_{j=0}^{\infty}\frac{(j+1)\ldots(j+m-1)}{(m-1)}Z^j. \tag{3.8}$$

Thus to calculate S_m we need to calculate $C_{j,m}$ only for j in multiples of $(2n+1)$ from (3.8). For $m = 5,6$

(3.9)

$$(2n+1)^4\sum_{k=-n}^{n} l^5_{kn}(t) = C_{0,5} + 2[C_{2n+1,5}\cos(2n+1)t + C_{4n+2,5}\cos(4n+2)t],$$

(3.10)

$$(2n+1)^5\sum_{k=-n}^{n} l^6_{kn}(t) = C_{0,6} + 2[C_{2n+1,6}\cos(2n+1)t + C_{4n+2,6}\cos(4n+2)t].$$

The numbers $C_{4n+2,5}$, $C_{2n+1,5}$ and $C_{0,5}$ are the coefficients of Z^{4n+2}, Z^{2n+1} and Z_0, respectively in the expansion (7) for $m = 5$. Thus

$$C_{4n+2,5} = \frac{1}{4!}[(9n+3)(9n+4)(9n+5)(9n+6) - 5(7n+2)(7n+3)(7n+4)(7n+5) + 10(5n+1)(5n+2)(5n+3)(5n+4) - 10(3n)(3n+1)(3n+2)(3n+3) + 5(n-1)n(n+1)(n+2)] = \frac{1}{4!}(n^4 + 2n^3 - n^2 - 2n)$$

$$C_{2n+1,5} = \frac{1}{4!}[(7n+2)(7n+3)(7n+4)(7n+5) - 5(5n+1)(5n+2)(5n+3)(5n+4) + 10(3n)(3n+1)(3n+2)(3n+3) - 10(n-1)n(n+1)(n+2)] = \frac{1}{4!}(76n^4 + 152n^3 + 104n^2 + 28n)$$

$$C_{0,5} = \frac{1}{4!}[(5n+1)(5n+2)(5n+3)(5n+4) - 5(3n)(3n+1)(3n+2)(3n+3) + 10(n-1)n(n+1)(n+2)] = \frac{1}{4!}(230n^4 + 460n^3 + 370n^2 + 140n + 24)$$

Similarly the coefficients of Z^{4n+2}, Z^{2n+1} and the constant term in (3.8) for $m = 6$ give

$$C_{4n+2,6} = \frac{1}{5!}\,[(10n+3)(10n+4)(10n+5)(10n+6)(10n+7) -$$

$$- 6(8n+2)(8n+3)(8n+4)(8n+5)(8n+6) +$$

$$+ 15(6n+1)(6n+2)(6n+3)(6n+4)(6n+5) -$$

$$- 20(4n)(4n+1)(4n+2)(4n+3)(4n+4) +$$

$$+ 15(2n-1)(2n)(2n+1)(2n+2)(2n+3)] =$$

$$= \frac{2n+1}{5!}\,(16n^4 + 32n^3 + 4n^2 - 12\ n).$$

$$C_{2n+1,6} = \frac{1}{5!}\,[(8n+2)(8n+3)(8n+4)(8n+5)(8n+6) -$$

$$- 6(6n+1)(6n+2)(6n+3)(6n+4)(6n+5) +$$

$$+ 15(4n)(4n+1)(4n+2)(4n+3)(4n+4) -$$

$$- 20(2n-1)(2n)(2n^3+1)(2n+2)(2n+3)] =$$

$$= \frac{2n+1}{5!}\,(416n^4 + 832n^3 + 584n^2 + 168n)$$

$$C_{0,6} = \frac{1}{5!}\,[(6n+1)(6n+2)(6n+3)(6n+4)(6n+5) -$$

$$- 6(4n)(4n+1)(4n+2)(4n+3)(4n+4) +$$

$$+ 15(2n-1)(2n)(2n+1)(2n+2)(2n+3)] =$$

$$= \frac{2n+1}{5!}\,(1056n^4 + 2112n^3 + 1704n^2 + 684n + 120)$$

Hence on simplifying the expressions for these numbers and putting in (3.9) and (3.10) we get the required sums S_5 and S_6 in the lemma.

4. The main lemma. An important property of the sum of the polynomials $v_k(x)$ [3] which plays a vital role in these investigations, is contained in the following lemma 2. We mention here that we have not been able to find an identity similar to (3.6) of Tureckii in the powers of $l_k(t)$ higher than 4 so as to have the identity

$$\sum_{k=0}^{n} v_k(x) = 1$$

as in the Kis-Vertesi operator.

[3] For typographical reasons we shall be writing k instead of kn later on.

Lemma 2. *For* $-1 \leq x \leq 1$,

(a) $$\left|\sum_{k=0}^{n} v_k(x) - 1\right| \leq \frac{1}{n^3},$$

(b) $$\left|\sum_{k=0}^{n} v_k'(x)\right| \leq 3.$$

Proof. From (2.3), (2.5) and (3.6), we have

$$\sum_{k=0}^{n} v_k(x) = \sum_{k=-n}^{n} u_k(t) = 1 + \frac{10\,n(n+1)}{9\,(2n+1)^4}\,[3 - 4\cos(2n+1)t + \cos(4n+2)t]$$

from which we at once have the part (a) of the lemma.
Further, on differentiation, we have

$$\sum_{k=0}^{n} v_k'(x) = \sum_{k=-n}^{n} \frac{u_k'(t)}{\sin t} = \frac{20n(n+1)\,[2\sin(2n+1)t - \sin(4n+2)t]}{9(2n+1)^3\cdot\sin t}$$

$$= \frac{40n(n+1)(2n+1)\sin(2n+1)t}{9(2n+1)^4\sin t} - \frac{20n(n+1)(2n+1)\sin(4n+2)t}{9(2n+1)^4\sin t}$$

Now using

(4.1) $$|\sin nt| \leq n|\sin t|, \text{ we have}$$

$$\left|\sum_{k=0}^{n} v_k'(x)\right| \leq 3.$$

5. Convergence of operators $V_{nr}(f, x)$.

The following theorem gives the convergence behaviour of the operators $V_{nr}\ (f, x)$.

THEOREM 2. *Let* $f(x) \in C^r[-1,1]$, *then for the operators* $V_{nr}\ (f,x)$, *we have*

(5.1) $$||V_{nr}(f,x) - f(x)|| \leq \frac{C_1^{4)}}{n^r}\,\omega_{f^{(r)}}\left(\frac{1}{n}\right), \qquad r = 0,1$$

where $\omega_{f^{(r)}}$ (.) *is the modulus of continuity of* $f^{(r)}$ *and* $||.|| = \max\limits_{-1\leq x\leq 1}|.|$ *is the uniform norm.*

For $V_{n0}(f,x)$ a stronger estimate than (5.1) holds, namely, the following theorem is valid.

[4] In the sequel we shall denote $C_1, C_2, \ldots$ for eight arbitrary constants.

THEOREM 3. *Let $f(x) \in C[-1,1]$, then for the operators $V_{n0}(f,x)$, we have,*

$$(5.2)\qquad |V_{n0}(f, x) - f(x)| \leq C_2\, \omega_f\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)$$

and

$$(5.3)\qquad |V_{n0}(f, x)| \leq C_3\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{-1} \omega_f\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right), \quad -1 \leq x \leq 1.$$

(5.2) is the well-known inequality of A.F. Timan. By proving (5.3) we are able to give a new proof of R. M. Trigub's inequality through interpolation.

For $V_{n1}(f,x)$ the inequality of A. F. Timan does not appear to hold. However, a better local estimate can be obtained but it is of no interest to us here, so we confine ourselves to proving only (5.1) for $r = 1$.

Proof of the theorem 2($r = 1$). On account of (2.4) we have the identity

$$(5.4)\qquad V_{n1}(f, x) - f(x) = [f(x) - f(-1) - (x+1)f'(-1)]\left[\sum_{k=0}^{n} v_k(x) - 1\right] -$$

$$- \sum_{k=0}^{n} [f(x) - f(x_k) - (x - x_k) f'(x_k)]\, v_k(x) = \sum\nolimits_{11} + \sum\nolimits_{12}.$$

Now making use of the well-known relation

$$(5.5)\qquad f(u) - f(v) - (u - v)\, f'(v) = O(|u - v|\, \omega_{f'}(|u - v|)),$$

we have

$$\left|\sum\nolimits_{11}\right| \leq C_4(x+1)\, \omega_{f'}(x+1)\left|\sum_{k=0}^{n} v_k(x) - 1\right|,$$

which by virtue of the identity in the part (a) of the lemma 2, gives

$$(5.6)\qquad \left|\sum\nolimits_{11}\right| \leq \frac{2C_4}{n^2}\, \omega_{f'}(2) \leq \frac{8C_4}{n}\, \omega_{f'}\left(\frac{1}{n}\right).$$

Replacing x by $\cos t$ and x_k by $\cos t_k$ and using (2.5) we can write

$$\sum\nolimits_{12} = \sum_{k=-n}^{n} [f(\cos t) - f(\cos t_k) - (\cos t - \cos t_k) f'(\cos t_k)]\; u_k(t).$$

Denote by t_j the nearest node to t, i.e. let

$$(5.7)\qquad |t - t_j| \leq \frac{\pi}{2n+1}.$$

Since the functions involved in the expression for $\sum_{12}$ do not change if we increase or decrease the numbers k in multiples of $(2n + 1)$, we have

$$(5.8)\qquad \sum\nolimits_{12} = \sum_{k=j-n}^{j+n} [f(\cos t) - f(\cos t_k) - (\cos t - \cos t_k) f'(\cos t_k)]\; u_k(t).$$

Because of (5.5) and the property of modulus of continuity, we have

$$\left|\sum_{12}\right| \leq C_5 \sum_{k=j-n}^{j+n} |\cos t - \cos t_k|\; \omega_{f'}(|\cos t - \cos t_k|)\; u_k(t) \leq$$

$$\leq C_5\, \omega_{f'}\left(\frac{1}{n}\right) \sum_{k=j-n}^{j+n} n(\cos t - \cos t_k)^2 + |\cos t - \cos t_k|\, |u_k(t)| \leq$$

$$\leq C_5\, \omega_{f'}\left(\frac{1}{n}\right) \sum_{k=j-n}^{j+n} \left[4n \sin^2\frac{t - t_k}{2} + 2 \sin\frac{|t - t_k|}{2}\right] |u_k(t)|,$$

which on using the estimate in part (a) of lemma 3 (§6), gives

$$(5.9)\qquad \left|\sum\nolimits_{12}\right| \leq \frac{C_6}{n}\, \omega_{f'}\left(\frac{1}{n}\right).$$

Thus (5.4), (5.6) and (5.9) prove theorem 2 for $r = 1$.

Proof of theorem 3. On account of (2.4), we have

$$(5.10)\qquad V_{n0}(f, x) - f(x) = [f(x) - f(-1)]\left[\sum_{k=0}^{n} v_k(x) - 1\right] -$$

$$- \sum_{k=0}^{n} [f(x) - f(x_k)]\, v_k(x) = \sum\nolimits_{01} + \sum\nolimits_{02}.$$

The sum $\sum_{01}$ in absolute value is

$$(5.11)\qquad \left|\sum\nolimits_{01}\right| \leq \frac{1}{n^2}\, \omega_f(x+1) \leq 4\, \omega_f\left(\frac{1}{n^2}\right).$$

Similar to the arguments leading to (5.8) we have

$$\sum\nolimits_{02} = \sum_{k=j-n}^{j+n} [f(\cos t) - f(\cos t_k)]\; u_k(t).$$

We shall now use the following well-known and easy to verify the relation

$$(5.12)\qquad \omega_f(|\cos t - \cos t_k|) \leq \left(2n \sin\frac{t - t_k}{2} + 1\right) \omega_f\left(\frac{\sin t}{n}\right) +$$

$$+ \left(2n^2 \sin^2\frac{t - t_k}{2} + 1\right) \omega_f\left(\frac{1}{n^2}\right).$$

Hence we have

$$\left|\sum\nolimits_{02}\right| \leqslant \omega_f\left(\sin\frac{t}{n}\right)\left\{\sum_{k=-n}^{n}\left(2n\sin\frac{t-t_k}{2}+1\right)|u_k(t)|\right\}+$$

$$+\omega_f\left(\frac{1}{n^2}\right)\left\{\sum_{k=-n}^{n}\left(2n^2\sin^2\frac{t-t_k}{2}+1\right)|u_k(t)|\right\},$$

again using part (a) of lemma 3, we have

(5.13) $$\left|\sum\nolimits_{02}\right| \leqslant C_6\left[\omega_f\left(\sin\frac{t}{n}\right)+\omega_f\left(\frac{1}{n^2}\right)\right].$$

Thus from (5.10), (5.11) and (5.13), we have the first part of the theorem 3.

To prove the second part of the theorem, we have from (2.4) on differentiation

(5.14) $$V'_{n0}(f, x) = \sum_{k=0}^{n}(f(x_k) - f(-1))\, v'_k(x) =$$

$$= \sum_{k=0}^{n}(f(x_k) - f(x))\, v'_k(x) + \sum_{k=0}^{n}(f(x) - f(-1))\, v'_k(x) = \sum\nolimits'_{01} + \sum\nolimits'_{02}.$$

Now on account of the estimate in part (b) of the lemma 2, we have

(5.15) $$\left|\sum\nolimits'_{02}\right| \leqslant 3\ \omega_f(x+1) \leqslant 6\ \omega_f(1) \leqslant \frac{12\,\omega_f\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)}{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)}.$$

Further we have

$$\sum_{k=0}^{n}[f(x_k) - f(x)]\, v'_k(x) = \sum_{k=j-n}^{j+n}[f(\cos t_k) - f(\cos t)]\frac{u'_k(t)}{\sin t}.$$

Hence

$$\left|\sum\nolimits'_{01}\right| \leqslant \omega_f\left(\frac{\sqrt{1-x^2}}{n}\right)\left[\frac{1}{\sin t}\sum_{k=j-n}^{j+n}\left(2n\sin\frac{|t-t_k|}{n}+1\right)u'_k(t)\right]+$$

$$+\ \omega_f\left(\frac{1}{n^2}\right)\left[\frac{1}{\sin t}\sum_{k=j-n}^{j+n}\left(2n^2\sin^2\frac{t-t_k}{2}+1\right)u'_k(t)\right]$$

and using the estimates in part (c) of lemma 3, we have

(5.16)

$$\left|\sum\nolimits'_{01}\right| \leqslant C_7\frac{n}{\sin t}\,\omega_f\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right) \leqslant 2C_7\frac{\omega_f\left(\frac{\sqrt{1-x^2}}{2}+\frac{1}{n^2}\right)}{\left(\frac{\sqrt{1-x^2}}{2}+\frac{1}{n^2}\right)} \text{ if } \sqrt{1-x^2} > \frac{1}{n}.$$

If $\sqrt{1-x^2} \leqslant \frac{1}{n}$, then we make use of part (b) of lemma 3, and we have

(5.17) $$\left|\sum\nolimits'_{01}\right| \leqslant C_8\, n^2\,\omega_f\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right) \leqslant 2C_8\frac{\omega_f\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)}{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)}.$$

Thus from (5.15), (5.16) and (5.17), we get the second part of the theorem.

6. An auxiliary lemma.

Lemma 3. *For* $t \in [0, \pi]$, *we have for* $p = 0, 1, 2$

(a) $$\sum_{k=j-n}^{j+n}\sin^p\frac{|t-t_k|}{2}\,|u_k(t)| \leqslant \frac{1201}{n^p}$$

(b) $$\sum_{k=j+n}^{j+n}\sin^p\frac{|t-t_k|}{2}\,|u'_k(t)| \leqslant \frac{1707}{n^{p-1}}$$

(c) $$\sum_{k=j-n}^{j+n}\sin^p\frac{|t-t_k|}{2}\,\frac{|u'_k(t)|}{\sin t} \leqslant \frac{1707}{n^{p-2}}.$$

Proof. We have

(6.1) $$\sum_{k=j-n}^{j+n}\sin^p\frac{|t-t_k|}{2}\,|u_k(t)| = \sin^p\frac{|t-t_j|}{2}\,|u_j(t)| + \sum_{\substack{k=j-n\\k\neq j}}^{j+n}\sin^p\frac{|t-t_k|}{2}\,|u_k(t)|.$$

Now, from (2.3) and (4.3), we have

(6.2) $$|u_k(t)| \leqslant \frac{67}{3}\,t_k^4(t)$$

and

(6.3) $$\sin^p\frac{|t-t_k|}{2}\,|l_k^p(t)| \leqslant \frac{1}{(2n+1)^p} \text{ for } k \neq j.$$

Let us define

(6.4) $$s_k = (2n+1)\sin\frac{t-t_k}{2}$$

then, by using

$$|t - t_k| \leqslant \frac{2|k-j| - 1}{2n+1} \pi$$

we have

(6.5) $$|s_k| \leqslant 2i - 1,$$

where

$$i = |k - j|, (k = j \pm 1, j \pm 2, \ldots, j \pm n).$$

Hence, on combining (6.1), (6.2), (6.3) and (6.5) we get

$$\sum_{k=j-n}^{j+n} \sin^p \frac{|t-t_k|}{2} |u_k(t)| \leqslant$$

$$\leqslant \frac{67}{3} \frac{\pi^p}{(2n+1)^p} + \frac{67}{3(2n+1)^p} \sum_{i=1}^{\infty} \frac{1}{(2i-1)^{4-p}} \leqslant$$

$$\leqslant \frac{120}{n^p}, \text{ for } p = 0, 1, 2.$$

which proves the part (a).
Now differentiating the formula (2.2), we get

$$l_k'(t) = \frac{\cos \frac{2n+1}{2}(t-t_k)}{2 \sin \frac{1}{2}(t-t_k)} - \frac{1}{2} \frac{\sin \frac{2n+1}{2}(t-t_k) \cos \frac{1}{2}(t-t_k)}{(2n+1)\sin^2 \frac{1}{2}(t-t_k)}.$$

Hence, we have, on using (6.4)

(6.9) $$|l_k'(t)| \leqslant \frac{2n+1}{2} \frac{1}{|s_k|}.$$

Now making use of (2.3) after differentiation and (6.3), we have

$$\sum_{k=j-n}^{j+n} \sin^p \frac{|t-t_k|}{2} |u_k'(t)| \leqslant \frac{160}{3} \frac{\pi^p}{(2n+1)^{p-1}} + \frac{160}{3(2n+1)^{p-1}} \sum_{\substack{k=j-n \\ k \neq j}}^{j+n} \frac{1}{|s_k|^{4-p}} \leqslant$$

$$\leqslant \frac{160}{3} \frac{\pi^p}{(2n+1)^{p-1}} + \frac{160}{3(2n+1)^{p-1}} \sum_{i=1}^{\infty} \frac{1}{(2i-1)^{4-p}} \leqslant \frac{1707}{n^{p-1}}, \text{ for } p = 0, 1, 2$$

from which we have (b).

In the similar fashion, we can prove (c) if we make use of (4.1).

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