MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 7, N° 2, 1978, pp. 135—139

ON THE METHOD OF THE CHORDS AND STEFFENSEN'S METHOD IN BOUNDED REGIONS IN BANACH SPACES

by
M. BALÁZS
(Cluj-Napoca)

Let be $F : X \to X$ an operator and the equation

(1) $$x = F(x),$$

where X is a Banach space. We note with x^* a fixed point of the operator F, which, evidently, is also a solution of the equation (1). In our paper [2] we have studied a class of iterative methods for solving equation (1), in which the sequence (x_n), converging to x^* is given by the following formula :

(2) $$x_{n+1} = x_n - \{I - [u_n, \ v_n ; \ F]\}^{-1}[x_n - F(x_n)]$$

$(n = 0, 1, 2, \ldots)$, where $[u, v ; F]$ is a divided difference of the operator F in the point $(u, v) \in X \times X$, I denotes the identity operator, an x_0 and the sequences (u_n), (v_n) are suitably chosen. In the quoted paper [2] the convergence of the sequence (x_n) has been proved (see Theorems 1 and 3) in case when the divided difference $[\cdot, \cdot ; F]$ was uniformly bounded, respectively verified a Lipschitz-type condition in the entire space X. It is known that the same type of results can be established if we assume that the conditions are verified only in a ball which will contain x^* and the sequences (x_n), (y_n). In this paper we shall give and proof theorems, analogous to the theorems 1 and 3 from the paper [2], when the conditions of these theorms are satisfied only in a bounded region from X.

THEOREM 1. *If*

(i.1) $$\|[u, v ; F]\| \leqslant \alpha < \frac{1}{3}$$

for all u, v from the ball

$$B_\alpha = \left\{x : \|x - x_0\| \leqslant \frac{1+\alpha}{(1-\alpha)^2}\|x_0 - F(x_0)\|\right\},$$

then the sequence (x_n) *defined by the formula* (2) *converges to the unique solution* x^* *of the equation* (1) *from the ball* B_α *for each sequence* (u_n) *and* (v_n) *from the ball* B_α *and the rate of convergence of the sequence* (x_n) *is given by the following inequality:*

$$\text{(j.1)} \qquad \|x^* - x_n\| \leqslant \left(\frac{2\alpha}{1-\alpha}\right)^n \frac{\|x_0 - F(x_0)\|}{1-\alpha}.$$

Proof. The existence and the unicity of x^* in B_α results from the contraction mapping principle. Really, from the condition (i.1) for any $x, y \in B_\alpha$ we have

$$\|F(x) - F(y)\| = \|[x, y; F](x-y)\| \leqslant \|[x, y; F]\| \cdot \|x - y\| \leqslant \alpha\|x - y\|.$$

According to the same principle it results

$$\text{(3)} \qquad \|x^* - x_0\| \leqslant \frac{\|x_0 - F(x_0)\|}{1-\alpha}.$$

From (i.1) results that for any $u_n, v_n \in B_\alpha$ the operator $I - [u_n, v_n; F]$ has an inverse and

$$\text{(4)} \qquad \|\{I - [u_n, v_n; F]\}^{-1}\| \leqslant \frac{1}{1-\alpha}.$$

Form (2) when $n = 0$, by (4) results

$$\|x_1 - x_0\| \leqslant \frac{\|x_0 - F(x_0)\|}{1-\alpha} < \frac{1+\alpha}{(1-\alpha)^2}\|x_0 - F(x_0)\|,$$

and thus $x_1 \in B_\alpha$.

Let's assume that $x_k \in B_\alpha$ for any $k \leqslant m$, where $m > 1$ is a positive integer. From (2) using the equality $x^* = F(x^*)$ and the evident identity

$$y - \{I - [u_k, v_k; F]\}^{-1} y = -\{I - [u_k, v_k; F]\}^{-1}[u_k, v_k; F]y$$

($y \in X$) results

$$\text{(5)} \quad x_{k+1} - x^* = \{I - [u_k, v_k; F]\}^{-1}\{F(x_k) - F(x^*) - [u_k, v_k; F](x_k - x^*)\}.$$

Because $x^*, x_k \in B_\alpha (k \leqslant m)$ we obtain

$$\|F(x_k) - F(x^*)\| \leqslant \|[x_k, x^*; F]\| \cdot \|x_k - x^*\| \leqslant \alpha\|x_k - x^*\|$$

and so

$$\text{(6)} \quad \|x_{k+1} - x^*\| \leqslant \frac{1}{1-\alpha}(\alpha\|x_k - x^*\| + \alpha\|x_k - x^*\|) = \frac{2\alpha}{1-\alpha}\|x_k - x^*\|.$$

If we put $k = 0, 1, \ldots, m$ in the inequality (6) and we multiply the inequalities obtained this way, we come to

$$\|x_{m+1} - x^*\| \leqslant \left(\frac{2\alpha}{1-\alpha}\right)^{m+1}\|x_0 - x^*\|,$$

whence, using (3), results

$$\text{(7)} \qquad \|x_{m+1} - x^*\| \leqslant \left(\frac{2\alpha}{1-\alpha}\right)^{m+1}\frac{\|x_0 - F(x_0)\|}{1-\alpha}$$

and so (j.1) is true for $n = m + 1$.

Applying the triangle inequality and using (3) and (7) we obtain

$$\|x_{m+1} - x_0\| \leqslant \|x_{m+1} - x^*\| + \|x^* - x_0\| \leqslant \frac{\|x_0 - F(x_0)\|}{1-\alpha}\left[1 + \left(\frac{2\alpha}{1-\alpha}\right)^{m+1}\right] \leqslant$$

$$\leqslant \frac{\|x_0 - F(x_0)\|}{1-\alpha}\left(1 + \frac{2\alpha}{1-\alpha}\right) = \frac{1+\alpha}{(1-\alpha)^2}\|x_0 - F(x_0)\|,$$

which shows that $x_{m+1} \in B_\alpha$. So, on the basses of the mathematical induction, $x_n \in B_\alpha$ for each positive integer n. The mathematical induction together with (7) also leads us to (j.1) for any positive integer n.

The convergence of the sequence (x_n) to x^* is guaranteed by the condition (i.1).

THEOREM 2. *If for any x, u, v from the ball*

$$B = \left\{x : \|x - x_0\| \leqslant \frac{2\|x_0 - F(x_0)\|}{1-\alpha}\right\}$$

the following conditions are verified:

$$\text{(i.2)} \qquad \|[u, v; F]\| \leqslant \alpha < 1$$

$$\text{(ii.2)} \qquad \|[x, u; F] - [x, v; F]\| \leqslant K\|u - v\|$$

and the equality

$$h_0 = \frac{K\alpha}{(1-\alpha)^2}\|x_0 - F(x_0)\| < 1$$

is true, then Steffensen's method, given by

$$\text{(8)} \qquad x_{n+1} = x_n - \{I - [x_n, F(x_n); F]\}^{-1}[x_n - F(x_n)]$$

converges to the unique solution x^*, *from the ball B, of the equation* (1) *and the rate of convergence is given by the following inequality:*

$$\text{(j.2)} \qquad \|x_n - x^*\| \leqslant (h_0)^{2^n - 1}\frac{\|x_0 - F(x_0)\|}{1-\alpha}.$$

Proof. The existence and uniqueness of the solution x^* in B results from the contraction mapping principle.

We shall proof that the terms of the sequences (x_n) and $(F(x_n))$ belong to the ball B. Evidently $F(x_0) \in B$.

Thus $\|[x_0, F(x_0); F]\| \leqslant \alpha < 1$ and so the inverse of the operator $I - [x_0, F(x_0); F]$ exists and we have

$$\|[I - [x_0, F(x_0); F]\}^{-1}\| \leqslant \frac{1}{1-\alpha}.$$

From (8) when $n = 0$ we get

$$\|x_1 - x_0\| \leqslant \frac{1}{1-\alpha}\|x_0 - F(x_0)\| \leqslant \frac{2\|x_0 - F(x_0)\|}{1-\alpha},$$

therefore $x_1 \in B$.

Let's assume that $x_k \in B$, $F(x_{k-1}) \in B$ for any $k \leqslant m$, where $m > 1$ is a positive integer. From (3) results that $x^* \in B$ too. Thus we have

$$\|x^* - F(x_m)\| = \|F(x^*) - F(x_m)\| \leqslant \|[x^*, x_m; F]\| \cdot \|x^* - x_m\| \leqslant \leqslant \alpha\|x^* - x_m\|. \tag{9}$$

From (5) when $u_k = x_k$ and $v_k = F(x_k)$ follows

$$x_{k+1} - x^* = \{I - [x_k, F(x_k); F]\}^{-1}\{F(x_k) - F(x^*) - [x_k, F(x_k); F](x_k - x^*)$$

which by (i.2), (ii.2) and (9), leads us to

$$\|x_{k+1} - x^*\| \leqslant \frac{1}{1-\alpha}\|[x_k, x^*; F] - [x_k, F(x_k); F]\| \cdot \|x_k - x^*\| \leqslant \tag{10}$$

$$\leqslant \frac{K}{1-\alpha}\|x^* - F(x_k)\| \cdot \|x_k - x^*\| \leqslant \frac{K\alpha}{1-\alpha}\|x^* - x_k\|^2.$$

(10) ensures the truth of (j.2) for each $k \leqslant n + 1$. The inequality (9), using (j.2) gives

$$\|x^* - F(x_m)\| \leqslant \alpha(h_0)^{2^m-1}\frac{\|x_0 - F(x_0)\|}{1-\alpha}. \tag{11}$$

Applying the triangle inequality, by (3) and (11) we obtain

$$\|x_0 - F(x_m)\| \leqslant \|x_0 - x^*\| + \|x^* - F(x_m)\| \leqslant \frac{\|x_0 - F(x_0)\|}{1-\alpha}[1 + \alpha(h_0)^{2^m-1}] <$$

$$< \frac{2\|x_0 - F(x_0)\|}{1-\alpha},$$

which means that $F(x_m) \in B$.

Further we have

$$\|x_{m+1} - x_0\| \leqslant \|x_{m+1} - x^*\| + \|x^* - x_0\| \leqslant \frac{\|x_0 - F(x_0)\|}{1-\alpha}[1 + (h_0)^{2^{m+1}-1}] <$$

$$< \frac{2\|(x_0 - F(x_0)\|}{1-\alpha},$$

which resulted using (3) and (10) for $k = m$. So $x_{m+1} \in B$. From the principle of the mathematical induction we obtain that x_n and $F(x_n)$ belong to B for each n. Because of the same principle (j.2) is true for any positive integer n.

The convergence of the sequence (x_n) to x^* is guaranteed by the condition $h_0 < 1$.

REFERENCES

[1] Balázs, M., *Contribution to the Study of Solving the Equations in Banach Spaces.* Doctor Thesis, Cluj-Napoca (1969).

[2] Balázs, M., *Notes on the Convergence of the Method of Chords and of Steffensen's Method in Banach Spaces* (in press).

[3] Rall, I. B., *Convergence of Stirling's Method in Banach Spaces.* Aequationes Mathematicae, **12**, *1.* 12–20 (1975).

[4] Sergeev, A. S., *On the Method of the Chords* (Russian). Sibirs. Mat. J. II, **2**, 282–289 (1961).

[5] Ul'm, S., *On Algorithm for Steffensen's Generalized Method* (Russian). Izv. Acad. Nauk Estonskoi S.S.R., **1**, 24–30 (1963).

Received 2.VI.1978.

Universitatea Babeş–Bolyai
Facultatea de matematică,
Str. Kogălniceanu 1
3400 Cluj - Napoca