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ANTIPROXIMINAL SETS IN BANACH SPACES OF c_0 — TYPE

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Let X be a normed linear space, M a nonvoid subset of X and x an element of X. Put $d(x, M) = \inf \{\| x - y \| : y \in M\}$ and $P_M(x) = \{y \in M : \| x - y \| = d(x, M\}$. The set M is called *proximinal* if $P_M(x) \neq \emptyset$, for all $x \in X$, and *antiproximinal* if $P_M(x) = \emptyset$, for all $x \in X \setminus M$. Following V. KLEE [6], a normed linear space is called of N_1 — *type* if contains a nonvoid closed convex antiproximinal set and of N_2 — *type*, if contains a nonvoid closed bounded convex antiproximinal set. Evidently, $N_2 \subset N_1$ and as was observed by V. KLEE [6], a Banach space is of N_1 — type if and only if is non-reflexive. The first example of a Banach space of N_2 — type was given by M. EDELSTEIN and A. C. THOMPSON [5]: the apce c_0 is of N_2 — type. In [2] it was shown that the space c is also of N_2 — type. These two results were extended in [3] in the following way: If S is a compact Hausdorff space such that $C(S)$ is isomorphic to c_0 and if J is an infinite dimensional closed ideal in $C(S)$, then J is of N_2 — type. (Here $C(S)$ denotes, as usually, the Banach space of all real-valued continuous functions on the Hausdorff compact S, with the supremum norm). In all of these cases it was shown that the considered spaces contain symmetric antiproximinal convex bodies. (By a convex body we mean a bounded closed convex set with nonvoid interior). Concerning the space $L_1(S, \mathcal{A}, \mu)$ it was proved that if $(S, \mathcal{A}, \mu)$ is a countably additive positive measure space, containing at least one atom, then $L_1(S, \mathcal{A}, \mu)$ is not of N_2 — type [1].

In this Note we shall give a slight extension of the result from [5].

Let X_i, $i \in I$, be a family of Banach spaces. Denote by $\left(\sum_i X_i\right)_{c_0}$ the Banach space of all functions $x : I \to \bigcup_i X_i$, such that $x(i) \in X_i$, for all $i \in I$ and the set $\{i \in I : \| x(i) \| > \varepsilon\}$ is finite, for all $\varepsilon > 0$. The norm on $\left(\sum_i X_i\right)_{c_0}$ is given by

(1) $$\| x\| = \sup \{\| x(i) \| : i \in I\} = \max \{\| x(i) \| : i \in I\}.$$

Denote also by $\left(\sum_i X_i\right)_{l_1}$ the Banach space of all functions $x : I \to \bigcup_i X_i$, such that $x(i) \in X_i$, for all $i \in I$, and $\sum_i \| x(i) \| < \infty$. The norm on $\left(\sum_i X_i\right)_{l_1}$ is given by

(2) $$\| x \| = \sum_i \| x(i) \|, \; x \in \left(\sum_i X_i\right)_{l_1}.$$

If X is a Banach space, denote by $c_0(X)$ and $l_1(X)$ the spaces $\left(\sum_{i \in \mathbf{N}} X_i\right)_{c_0}$ respectively $\left(\sum_{i \in \mathbf{N}} X_i\right)_{l_1}$, where $X_i = X$ for all $i \in \mathbf{N}$. The value of a function x in one of these spaces at $i \in I$, will be denoted also by x_i.

If M is a convex subset of a locally convex space X, we say that a functional $f \in X'$ (X' — the conjugate of X) *supports* the set M if there exists $x_0 \in M$ such that $f(x_0) = \sup f(M)$ or $f(x_0) = \inf f(M)$. Denote by $\mathcal{S}(M)$ the set of all support functionals of the set M.

In the sequel we shall need the following two results from [5]:

Lemma A. *A closed convex subset C of a normed linear space X is antiproximinal if and only if*
$$\mathcal{S}(B) \cap \mathcal{S}(C) = \{0\},$$
where B denotes the closed unit ball of X.

Lemma B. *If X, Y are Banach spaces, C a convex subset of X and $A : X \to Y$ is an isomorphism, then*
$$\mathcal{S}(C) = A'\mathcal{S}(A(C)),$$
where $A' : Y' \to X'$ denotes the conjugate operator of A.

We can now state the result of this Note:

PROPOSITION *The space $X = \left(\sum_i X_i\right)_{c_0}$ is of N_2 — type for every infinite set I and every family X_i, $X_i \neq \{0\}$, $i \in I$, of Banach spaces. More exactly, X contains a symmetric antiproximinal convex body.*

Proof. Firstly, observe that $\left(\sum_i X_i\right)_{c_0}$ is isometrically isomorphic to $c_0\left(\left(\sum_i X_i\right)_{c_0}\right)$, for every infinite set I and every family $\{X_i : i \in I\}$ of Banach spaces. Indeed, since the set I is infinite it can be written as $I_n = \bigcup I_n$, where $I_n \cap I_m = \emptyset$, for $m \neq n$, and card (I_n) = card (I), for

all $n \in \mathbf{N}$. Let $\lambda_n : I_n \to I$, $n \in \mathbf{N}$, be some bijections. For $\xi \in c_0\left(\left(\sum_i X_i\right)_{c_0}\right)$ denote by $\xi_n \in \left(\sum_i X_i\right)_{c_0}$ the value of the function ξ at $n \in \mathbf{N}$, and by $\xi_n(i)$, the value of the function ξ_n at $i \in I$. Then an isometry $T : c_0\left(\left(\sum_i X_i\right)\right)_{c_0} \to \left(\sum_i X_i\right)_{c_0}$ is given by
$$(T\xi)(i) = \xi_n(\lambda_n(i)),$$
for all $i \in I_n$, $n \in \mathbf{N}$ and $\xi \in c_0\left(\left(\sum_i X_i\right)_{c_0}\right)$. The converse of T is given by
$$(T^{-1}x)_n(i) = x(\lambda_n^{-1}(i)),$$
for all $i \in I_n$, $n \in \mathbf{N}$ and $x \in \left(\sum_i X_i\right)_{c_0}$. Therefore it is sufficient to prove the Proposition in the case of a space $c_0(X)$, where X is a Banach space, $X \neq \{0\}$.

Suppose then, that X is a Banach space, $X \neq \{0\}$, and define $A : c_0(X) \to c_0(X)$, by

(3) $$Ax(i) = x(i) + \sum_{k=1}^{\infty} 2^{-(k+1)} x(\sigma_i(k)),$$

for $i \in \mathbf{N}$ and $x \in c_0(X)$, where $\sigma_n : \mathbf{N} \to \mathbf{N}$ is an application given by

(4) $$\sigma_n(k) = 2^{n-1}(2k - 1), \; k \in \mathbf{N},$$

for all $n \in \mathbf{N}$. Since the space X is complete and $\| x(i) \| + \sum_{k=1}^{\infty} 2^{-(k+1)} \| x(\sigma_i(k)) \| < \infty$, it follows that, for all $x \in c_0(X)$ and $i \in \mathbf{N}$, there exists $Ax(i) \in X$, such that (3) holds. Obviously, $\lim_{i\to\infty} Ax(i) = 0$, i.e. $Ax \in c_0(X)$ and the operator A is linear. The conjugate of $c_0(X)$ is the space $l_1(X')$, the duality between $c_0(X)$ and $l_1(X')$ being given by
$$f(x) = \sum_{i=1}^{\infty} f_i(x(i)),$$
for $x \in c_0(X)$ and $f \in l_1(X')$ (see [4]). Let $S_{X'} = \{x' \in X' : \|x'\| = 1\}$ be the unit sphere of X'. Define the isometric injections $\varphi_i : X \to c_0(X)$ and $\psi_i : X' \to l_1(X')$, by $\varphi_i x(j) = \delta_{ij} x$, respectively $(\psi_i x') = \delta_{ij} x'$, for all $i, j \in \mathbf{N}$, $x \in X$ and $x' \in X'$, where δ_{ij} denotes the Kronecker symbol. Let
$$B_1 = \{x \in c_0(X) : |\psi_i x'(Ax)| \leq 1 \text{ for all } x' \in S_{X'}, \text{ and } i \in \mathbf{N}\}.$$

Evidenthly, B_1 is a symmetric closed convex set set. Since, for $||x|| \leq 2/3$ we have

$$|(\psi_i x')Ax| = |x'(Ax(i))| \leq |x'(x(i))| + \sum_{k=1}^{\infty} 2^{-(k+1)}|x'(x(\sigma_i(k))| \leq ||x|| + \\ + 2^{-1}||x|| = (3/2)||x|| \leq 1,$$

for all $x' \in S_{X'}$, and $i \in \mathbf{N}$, it follows that 0 is an interior point of B_1.

We shall show that the set B_1 is also bounded. Suppose $x \in c_0(X)$ and $||x|| > 2$. Let $i_0 \in \mathbf{N}$ be such that $||x(i_0)|| = ||x||$ and let $x' \in S_{X'}$ be such that $x'(x(i_0)) = ||x(i_0)||$. Then

$$\psi_{i_0} x'(Ax) = x'(x(i_0)) + \sum_{k=1}^{\infty} 2^{-(k+1)}\, x'(x(\sigma_{i_0}(k)) = ||x(i_0)|| + \\ + \sum_{k=1}^{\infty} 2^{-(k+1)} x'(x(\sigma_{i_0}(k)) \geq ||x|| - 2^{-1}\, ||x|| > 1,$$

so that $x \notin B_1$, and therefore $||x|| \leq 2$ for all $x \in B_1$. Denoting by B the closed unit ball of $c_0(X)$ and, taking into account that, for all $x \in X$, $||x|| = \sup\, \{x'(x) : x' \in S_{X'}\}$ we get

$$B_1 = \{x \in c_0(X) : |\psi_i x'(Ax)| \leq 1,\ x' \in S_{X'},\ i \in \mathbf{N}\} = \\ = \{x \in c_0(X) : |x'(Ax(i))| \leq 1,\ x' \in S_{X'},\ i \in \mathbf{N}\} = \\ = \{x \in c_0(X) : ||Ax(i)|| \leq 1,\ i \in \mathbf{N}\} = \\ = \{x \in c_0(X) : ||Ax|| \leq 1\} = A^{-1}(B).$$

The equality $B_1 = A^{-1}(B)$ shows that A is an isomorphism of $c_0(X)$ onto $c_0(X)$ and, by Lemma B,

$$(5) \qquad \mathcal{S}(B_1) = A'\mathcal{S}(B).$$

Denoting by B_X the closed unit ball of X, observe that $f \in l_1(X')$ supports the unit ball of $c_0(X)$ if and only if there exists $n \in \mathbf{N}$ such that $f_i = 0$ for $i > n$ and $f_i \in \mathcal{S}(B_X)$, for $i = 1, 2, \ldots, n$. Now, if $f = (f_1, \ldots, f_n, 0, \ldots) \neq 0$ is a support functional of the unit ball of $c_0(X)$, then

$$A'f(x) = f(Ax) = \sum_{i=1}^{n} f_i(Ax(i)) = \\ = \sum_{i=1}^{n}\left(f_i(x(i)) + \sum_{k=1}^{\infty} 2^{-(k+1)} f_i x(\sigma_i(k))\right),$$

which shows that the set $\{i \in \mathbf{N} : (A'f)_i \neq 0\}$ is infinite and consequently $A'f \notin \mathcal{S}(B)$. Taking into account (5), it follows $\mathcal{S}(B) \cap \mathcal{S}(B_1) = \{0\}$ and therefore, by Lemma A, the set B_1 is antiproximinal, which ends the proof of the Proposition.

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