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MINIMAL MONOSPLINES IN L_2 AND OPTIMAL CUBATURE FORMULAE

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1. In [1] was introduced the two-dimensional monospline, namely; if $\Pi = \{(x_i, y_j) | 0 \leqslant x_0 < \ldots < x_m \leqslant 1;\ 0 \leqslant y_0 < \ldots < y_n \leqslant 1\}$ and S is a two-dimensional spline function of the degree $(r-1, s-1)$ with deficiency of order (μ, ν) and with the knots $(x_i, y_j) \in \Pi$, then the function

$$M(x, y) = \frac{x^r y^s}{r!s!} + S(x, y) \tag{1.1}$$

is called a two-dimensional monospline of the degree (r, s) with deficiency of order (μ, ν) and with the knots $(x_i, y_j) \in \Pi$. The set of these functions was denoted by $\mathfrak{M}_{rs}(m, n, \mu, \nu)$ or $\mathfrak{M}_{rs}$.

Furthermore, in [1] was considered the problem to find the monospline $\overline{M} \in \mathfrak{M}_{rs}(m, n, \mu, \nu)$ for which

$$\|\overline{M}\|_{L_2(D)} = \min_{M \in \mathfrak{M}_{rs}} \|M\|_{L_2(D)},$$

where $D = \{0 \leqslant x, y \leqslant 1\}$, and this problem was solved in the case $x_0 = y_0 = 0$, $x_m = y_n = 1$ and $\mu = r-1$, $\nu = s-1$.

Now, let $\mathfrak{M}^0_{rs}(m, n, r-1, s-1)$ be the set of those monosplines from $\mathfrak{M}_{rs}(m, n, r-1, s-1)$ for which

$$M^{(p,q)}(a, b) = 0,\ M^{(p,0)}(a, y) = 0,\ M^{(0,q)}(x, b) = 0,\ (a, b = 0,1;\ p = 0, \ldots, r-1;\ q = 0, \ldots, s-1). \tag{1.2}$$

One observes that the conditions (1,2) are satisfied if and only if

$$M^0(x, y) = \frac{x^r y^s}{r!s!} + \sum_{i,j=0}^{m,n} \sum_{p,q=0}^{r-1,s-1} \omega_{ij}^{pq} \frac{(x - x_i)_+^p}{p!} \frac{(y - y_j)_+^q}{q!} +$$

$$+ \frac{y^s}{s!} \sum_{i=0}^{m} \sum_{p=0}^{r-1} \alpha_{is}^p \frac{(x - x_i)_+^p}{p!} + \frac{x^r}{r!} \sum_{j=0}^{n} \sum_{q=0}^{s-1} \beta_{rj}^q \frac{(y - y_j)_+^q}{q!},$$

and

$$(1.3) \qquad \sum_{i,j=0}^{m,n} \sum_{p,q=0}^{r-1,s-1} \omega_{ij}^{pq} \frac{(x - x_i)_+^p}{p!} \frac{(y - y_j)_+^q}{q!} =$$

$$\sum_{i=0}^{m} \sum_{p=0}^{r-1} \alpha_{is}^p \frac{(x - x_i)_+^p}{p!} \sum_{j=0}^{n} \sum_{q=0}^{s-1} \beta_{rj}^q \frac{(y - y_j)_+^q}{q!}, \qquad x \in [x_m, 1] \quad y \in [y_n, 1]$$

$$\sum_{i=0}^{m} \sum_{p=0}^{r-1} \alpha_{is}^p \frac{(x - x_i)_+^p}{p!} = \frac{(x - 1)^r - x^r}{r!},$$

$$\sum_{j=0}^{n} \sum_{q=0}^{s-1} \beta_{rj}^q \frac{(y - y_j)_+^q}{q!} = \frac{(y - 1)^s - y^s}{s!}, \qquad x \in [x_m, 1] \quad y \in [y_n, 1].$$

for each $M^0 \in \mathfrak{M}_{rs}^0(m, n, r - 1, s - 1)$.

In this paper one considers the problem to find the monospline $\overline{M}^0 \in \mathfrak{M}_{rs}^0(m, n, r - 1, s - 1)$ which is of least deviation from zero in the square mean, on D.

The solution of this minimization problem is given by:

THEOREM 1. *If*

1) *m, n, r, s are given natural numbers,*

2) *$\mathfrak{M}_r^0$ and $\mathfrak{M}_s^0$ are respectively the set of one-dimensional monosplines of the degree r with deficiency of order $r - 1$ and with the knots $\{x_i\}$ and the set of one-dimensional monosplines of the degree s with dericiency of order $s - 1$ and with the knots $\{y_j\}$, which satisfy the conditions*

$$M_1^{(p)}(0) = M_1^{(p)}(1) = 0, \ (p = 0, \ \ldots, \ r - 1), \ \forall M_1 \in \mathfrak{M}_r^0$$

$$M_2^{(q)}(0) = M_2^{(q)}(1) = 0, \ (q = 0, \ \ldots, \ s - 1), \ \forall M_2 \in \mathfrak{M}_s^0$$

3) *$\overline{M}_1^0 \in \mathfrak{M}_r^0$ and $\overline{M}_2^0 \in \mathfrak{M}_s^0$ are respectively the monospines which are of least deviation from zero in the square mean, on $[0,1]$, then there exists a unique monospline $\overline{M}^0 \in \mathfrak{M}_{rs}^0$, which is of least deviation from zero in square mean, on D, namely*

$$\overline{M}^0(x, y) = \overline{M}_1^0(x) \ \overline{M}_2^0(y)$$

and

$$||\overline{M}^0||_{L_2(D)} = \frac{r!s!}{(2r + 1)(2s + 1)} \rho^r \eta^s,$$

where

$$\rho = \frac{1}{2\sqrt[r]{(r!)^2 + m\sqrt[r]{(2r)!}}}, \quad \eta = \frac{1}{2\sqrt[s]{(s!)^2 + n\sqrt[s]{(2s)!}}}.$$

Proof. It is known from [4] that in the set of the polynomials of the from

$$(1.4) \qquad P_{rs}(x, y) = x^r y^s + x^r \sum_{q=0}^{s-1} a_{rq} y^q + \sum_{p=0}^{r-1} a_{ps} x^p + \sum_{p,q=0}^{r-1,s-1} a_{pq} x^p y^q$$

the unique polynomial which is of least deviation from zero, on $[a - h \leqslant x \leqslant a + h;\ b - g \leqslant y \leqslant b + g]$, is

$$(1.5) \qquad \overline{P}_{rs}(x, y) = h^r g^s X_r\left(\frac{x - a}{h}\right) X_s\left(\frac{y - b}{g}\right),$$

where X_n is the Legendre polynominal of the degree n havind the coefficient of x^r equal with 1.

Thus, the proof proceeds by establishing the monospline $M_0 \in \mathfrak{M}_{rs}^0$ for which the integral

$$J = \int_0^1\int_0^1 [M_0(x, y)]^2 \, dxdy$$

takes the minimum value. This integral can be written

$$(1.6) \qquad J = \sum_{i,j=0}^{m+1,n+1} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} [M_{ij}^0(x, y)]^2 \, dxdy,$$

where $x_{-1} = y_{-1} = 0$, $x_{m+1} = y_{n+1} = 1$ and

$$(1.7) \qquad M_{ij}^0(x,y) = \frac{x^r y^s}{s!r!} + \sum_{k,l=0}^{i-1,j-1} \sum_{p,q=0}^{r-1,s-1} \omega_{kl}^{pq} \frac{(x - x_k)^p}{p!} \frac{(y - y_l)^q}{q!} +$$

$$+ \frac{y^s}{s!} \sum_{k=0}^{i-1} \sum_{p=0}^{r-1} \alpha_{ks}^p \frac{(x - x_k)^p}{p!} + \frac{x^r}{r!} \sum_{l=0}^{j-1} \sum_{q=0}^{s-1} \beta_{rl}^q \frac{(y - y_l)^q}{q!}.$$

Using the notations

$$\frac{\omega_{kl}^{pq}}{p!q!} = \frac{\theta_{kl}^{pq}}{r!s!}, \quad \frac{\alpha_{ks}^p}{p!} = \frac{\gamma_{ks}^p}{r!}, \quad \frac{\beta_{rl}^q}{q!} = \frac{\delta_{rl}^q}{s!},$$

it follows that

$$M_{ij}^0(x, y) = \frac{1}{r!s!} N_{ij}^0(x, y),$$

where

(1.8) $$N^0_{ij}(x, y) = x^r y^s + \sum_{k,l=0}^{i-1,j-1}\ \sum_{p,q=0}^{r-1,s-1} \theta^{pq}_{kl}(x - x_k)^p (y - y_l)^q +$$

$$+ y^s \sum_{k=0}^{i-1}\sum_{p=0}^{r-1} \gamma^p_{ks}(x - x_k)^p + x^r \sum_{l=0}^{j-1}\sum_{q=0}^{s-1} \delta^q_r (y - y_l)^q.$$

From (1.6) it follows

(1.9) $$J = \frac{1}{(r!)^2(s!)^2}\sum_{i,j=0}^{m+1,n+1} I_{ij},\quad I_{ij} = \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}[N^0_{ij}(x, y)]^2\,dxdy.$$

One observes that the intergral J takes its minimun value if and only if each of the integrals I_{ij} take the minimum value. But N^0_{ij}, $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ is a polynomial of the from (1.4). It follows that the integral I_{ij} takes its minimum value only if

(1.10) $$N^0_{ij}(x, y) = h^r_i g^s_j X_r\left(\frac{x - a_i}{h_i}\right) X_s\left(\frac{y - b_j}{g_j}\right),\quad (x, y) \in D_{ij},$$

where $D_{ij} = \{x_{i-1} \leqslant x \leqslant x_i;\ y_{j-1} \leqslant y \leqslant y_j\}$ and $a_i = \dfrac{x_{i-1} + x_i}{2}$,

$$h_i = \frac{x_i - x_{i-1}}{2},\quad b_j = \frac{y_{j-1} + y_j}{2},\quad g_j = \frac{y_j - y_{j-1}}{2}.$$

Assuming that the knots (x_i, y_j) are fiexd and taking into account that

$$\int_{a-h}^{a+h} X^2_n(x)dx = \frac{(n!)^4 2^{2n+1}}{(2n + 1)[(2n)!]^2}h^{2n+1},$$

we obtain

(1.11) $$I_{ij} = \frac{(r!)^4(s!)^4}{(2r + 1)[(2r)!]^2(2s + 1)[(2s)!]^2}(x_i - x_{i-1})^{2r+1}(y_j - y_{j-1})^{2s+1}$$

$(i = 1, \ldots, m;\ j = 1, \ldots, n)$.

From (1.3) it follows that

(1.12) $N^0_{ij}(x, y) = N^0_{1i}(x)\ N^0_{2j}(y),\ (x, y) \in D_{i0} \cup D_{i,n+1} \cup D_{0j} \cup D_{m+1,j}$, $(i = 0, 1, \ldots, m + 1;\ j = 0, 1, \ldots, n + 1)$, where

$$N^0_{1i}(x) = x^r + \sum_{k=0}^{i-1}\sum_{p=0}^{r-1}\gamma^p_{ks}(x - x_k)^p,\quad N^0_{1,m+1}(x) = (x - 1)^r$$

(1.13) $$N^0_{2j}(x) = y^s + \sum_{l=0}^{j-1}\sum_{q=0}^{s-1}\delta^q_{rl}(y - y_l)^q,\quad N^0_{2,n+1}(y) = (y - 1)^s.$$

From (1.9) and (1.11), taking into account that in the set of the from (1.13) the unique polynomials which, on $[x_{i-1}, x_i]$ respectively $[y_{j-1}, y_j]$, are of least deviation from zero in square mean, are the corresponding Legendre polynomials $h^r_i X_r\left(\frac{x - a_i}{h_i}\right)$ and $g^s_j X_s\left(\frac{y - b_j}{g_j}\right)$, it follows that

$$\bar{J} = \frac{1}{(r!)^2(s!)^2(2r + 1)(2s + 1)}\left[x_0^{2r+1} + \frac{(r!)^4}{[(2r)!]^2}\sum_{i=1}^{m}(x_i - x_{i-1})^{2r+1} +\right.$$

$$\left. + (1 - x_m)^{2r+1}\right]\cdot\left[y_0^{2s+1} + \frac{(s!)^4}{[(2s)!]^2}\sum_{j=1}^{n}(y - y_{j-1})^{2s+1} + (1 - y_n)^{2s+1}\right].$$

Now we shall minimize the function $\bar{J}$ with regard to the parameters x_i, y_j. It is easy to see that $\bar{J}$ takes its minimum value only when

(1.14) $$\bar{x}_0 = \rho\sqrt[r]{(r!)^2},\quad \bar{x}_1 = \bar{x}_0 + ih,\quad (i = 1, \ldots, m)$$

$$\bar{y}_0 = \eta\sqrt[s]{(s!)^2},\quad \bar{y}_j = \bar{y}_0 + jg,\quad (j = 1, \ldots, n)$$

where

(1.15) $$h = \rho\sqrt[r]{(2r)!},\quad g = \eta\sqrt[s]{(2s)!}$$

and

(1.16) $$\min_{x_i, y_j} \bar{J} = \frac{(r!)^2(s!)^2}{(2r + 1)(2s + 1)}\rho^{2r}\eta^{2s}.$$

Using again the identities (1.10), we shall determine the parameters θ^{pq}_{ij}. In the same way as in [1] we get

$$\bar{\omega}^{pq}_{ij} = \bar{\alpha}^p_i\bar{\beta}^q_j,\quad \bar{\alpha}^p_{is} = \bar{\alpha}^p_i,\quad \bar{\beta}^q_{rj} = \bar{\beta}^q_j,$$

where

(1.17) $$\bar{\alpha}^p_0 = \frac{\rho^{r-p}}{r!}\left\{(-1)^{r-p}X^{(p)}_r(1)[(2r)!]^{\frac{r-p}{2}} - \frac{r!}{(r - p)!}(r!)^{\frac{2(r-p)}{2}}\right\}$$

$$\bar{\alpha}^p_i = \frac{(-1)^{r-p} - 1}{r!}h^{r-p}X^{(p)}_r(1),\quad (i = 1, \ldots, m - 1)$$

$$\bar{\alpha}^p_m = (-1)^{r-p-1}\alpha^p_0,\quad (p = 0, 1, \ldots, r - 1),$$

and

$$\overline{\beta}_0^q = \frac{\eta^{s-q}}{s!}\left\{(-1)^{s-q}X_s(1)\left[(2s)!\right]^{\frac{s-q}{s}} - \frac{s!}{(s-q)!}(s!)^{\frac{2(s-q)}{2}}\right\},$$

$$\overline{\beta}_j^q = \frac{(-1)^{s-q}-1}{s!}g^{s-q}X_s^{(q)}(1), \quad (j = 1, \ldots, n-1) \tag{1.18}$$

$$\overline{\beta}_n^q = (-1)^{s-q-1}\overline{\beta}_0^q, \quad (q = 0, 1, \ldots, s-1).$$

From (1.8) it follows that

$$\overline{M}_{ij}^0(x, y) = \left[\frac{x^r}{r!} + \sum_{k=0}^{i-1}\sum_{p=0}^{r-1}\overline{\alpha}_k^p\frac{(x-\bar{x}_k)^p}{p!}\right]\left[\frac{y^s}{s!} + \sum_{l=0}^{j-1}\sum_{q=0}^{s-1}\overline{\beta}_l^q\frac{(y-\bar{y}_l)}{q!}\right].$$

Thus the theorem is proved.

2. Let us consider the problem to approximate the intergal

$$I = \int_0^1\int_0^1 f(x, y)\,dxdy$$

by a linear functional of the form

$$Q(f) = \sum_{i,j=0}^{m,n}\sum_{p,q=0}^{\mu,\nu}A_{ij}^{pq}f^{(p,q)}(x_i, y_j),$$

where A_{ij}^{pq}, x_i, y_j are real parameters satisfying the conditions:

$$0 \leqslant x_0 < \ldots < x_m \leqslant 1, \quad 0 \leqslant y_0 < \ldots < y_n \leqslant 1.$$

Let $M \in \mathfrak{M}_{rs}^0$. Because $M^{(r,s)} = 1$, we have

$$I = \int_0^1\int_0^1 M^{(r,s)}(x, y)f(x, y)\,dxdy,$$

or

$$I = \sum_{i,j=0}^{m,n}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} M_{ij}^{(r,s)}(x, y)f(x, y)\,dxdy, \tag{2.1}$$

where M_{ij} is a polynomial of the form (1.7). If we apply the general formula for integration by parts it follows that

$$\int_0^1\int_0^1 f(x, y)\,dxdy = \sum_{i,j=0}^{m,n}\sum_{p,q=0}^{r-1,s-1}(-1)^{p+q}[M^{(r-p-1,s-q-1)}(x_i - 0, y_j - 0) - \tag{2.2}$$
$$- M^{(r-p-1,s-q-1)}(x_i + 0, y_j - 0) - M^{(r-p-1,s-q-1)}(x_i - 0, y_j + 0) +$$
$$+ M^{(r-p-1,s-q-1)}(x_i + 0, y_j + 0)]f^{(p,q)}(x_i, y_j) + R(f),$$

where

$$R(f) = (-1)^r\int_0^1\int_0^1 M^{(0,s)}(x, y)f^{(r,0)}(x, y)\,dxdy + \tag{2.3}$$
$$+ (-1)^s\int_0^1\int_0^1 M^{(r,0)}(x, y)f^{(0,s)}(x, y)\,dxdy -$$
$$- (-1)^{r+s}\int_0^1\int_0^1 M(x, y)f^{(r,s)}(x, y)\,dxdy.$$

Thus we obtain a cubature formula with the coefficients

$$A_{ij}^{pq} = (-1)^{p+q}[M^{(r-p-1,s-q-1)}(x_i - 0, y_j - 0) - \tag{2.4}$$
$$- M^{(r-p-1,s-q-1)}(x_i + 0, y_j - 0) - M^{(r-p-1,s-q-1)}(x_i - 0, y_j + 0) +$$
$$+ M^{(r-p-1,s-q-1)}(x_i + 0, y_j + 0)]$$

and with the remainder (2.3). We shal denote the set of these cubature formulas by $\mathfrak{F}_{rs}$.

Assuming that $||f^{(r,0)}||_{L_2(D)} \leqslant P_{r0}$, $||f^{(0,s)}||_{L_2(D)} \leqslant P_{0s}$ and $||f^{(r,s)}||_{L_2(D)} \leqslant P_{rs}$, we obtain

$$|R(f)| \leqslant P_{r0}\left(\int_0^1\int_0^1[M^{(0,s)}(x, y)]^2\,dxdy\right)^{1/2} + \tag{2.5}$$
$$+ P_{0s}\left(\int_0^1\int_0^1[M^{(r,0)}(x, y)]^2\,dxdy\right)^{1/2} + P_{rs}\left(\int_0^1\int_0^1[M(x, y)]^2\,dxdy\right)^{1/2}.$$

One considers the following problem: if m, n, r and s are given natural numbers, determine $F \in \mathfrak{F}_{rs}$ for which the second member of (2.5), where P_{r0}, P_{0s}, P_{rs} are given real numbers, takes its minimum value.

We observe that the solution of this problem corresponds to the monospline $M \in \mathfrak{M}_{rs}$ for which the integrals

$$I_{0s} = \int_0^1\int_0^1[M^{(0,s)}(x, y)]^2\,dxdy, \quad I_{r0} = \int_0^1\int_0^1[M^{(r,0)}(x, y)]^2\,dxdy \tag{2.6}$$
$$I_{rs} = \int_0^1\int_0^1[M(x, y)]^2\,dxdy$$

take the minimum value.

Taking into account that

$$M^{(0,s)}(x,y) = \frac{x^r}{r!} - \sum_{i=0}^{m}\sum_{p=0}^{r-1} \alpha_{is}^p \frac{(x-x_i)_+^p}{p!} = M_1(x)$$

$$M^{(r,0)}(x,y) = \frac{y^s}{s!} - \sum_{j=0}^{n}\sum_{q=0}^{s-1} \beta_{rj}^q \frac{(y-y_j)_+^q}{q!} = M_2(y)$$

it follows (see theorem 1) that the integrals (2.6) take the minimum value only if:

$$M(x,y) = \overline{M}_1(x) \cdot \overline{M}_2(y)$$

where M_1 and M_2 are the one-dimensional monosplines of the degree r respectively s which, on [0,1], are of least deviation from zero in the square mean. These monosplines are defined by the values (1.14) respectively (1.17) — (1.18) of the parameters x_i, y_j and α_i^p, β_j^q.

From (2.4) one obtains

$$(2.7)\quad A_{ij}^{pq} = (-1)^{p+q-r-p-1-s-q-1} \overline{\alpha}_i \overline{\beta}_j, \quad \begin{pmatrix} i=0, \ldots, m; \; p=0, \ldots, r-1 \\ j=0, \ldots, n; \; q=0, \ldots, s-1 \end{pmatrix},$$

where $\overline{\alpha}_i^k$ and $\overline{\beta}_j^l$ are given by (1.17) respectively (1.18).

Also, from (1.16) and (2.5) it follows that

$$(2.8)\quad |R(f)| \leqslant P_{r0} \frac{r!}{\sqrt{2r+1}} \rho^r + P_{0s} \frac{s!}{\sqrt{2s+1}} \eta^s + P_{rs} \frac{r!s!}{\sqrt{(2r+1)(2s+1)}} \rho^r \eta^s$$

Thus we proved the following theorem:

THEOREM 2. *If m, n, r and s are given natural numbers, then in the set F_{rs} there exists a unique cubature formula for which the right member of the inequality* (2.5) *takes its minimum value. The coefficients of this formula are given by* (2.7), *the coordinate of knots have the expresions* (1.14) *and the remainder estimation is given by* (2.8).

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