MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 7, Nº 2, 1978, pp. 169–170

A CHARACTERIZATION OF SEMI-CHEBYSHEVIAN SETS IN A METRIC SPACE

by
COSTICĂ MUSTĂŢA
(Cluj-Napoca)

In [1] is given the theorems of characterization of Chebyshevian sets in a metric space. The present note is a completation of the paper [1].

Let X be a metric space with the metric d, let Y be a nonvoid subset of X such that $x_0 \in Y$, where x_0 is an fixed element in X. Let $\mathrm{Lip}_0 X$ be the space of all real Lipschitz functions, defined on X, endowed with the Lipschitz norm $\| \ \|_X$ [1].

The set Y is called *semi-Chebyshevian* if for every $x \in X \setminus Y$ there exists at most an element $y_0 \in Y$ such that

$$d(x, y_0) = \inf \{d(x, y) : y \in Y\} = d(x, Y). \tag{1}$$

An element $y_0 \in Y$ for that (1) holds is called an *element of best approximation* of x, by elements of Y.

THEOREM. *If Y is a nonvoid subset of the metric space X such that $x_0 \in Y$, then the following two asertions are equivalent:*

1° *Y is semi-Chebyshevian;*

2° *There does not exist $f \in \mathrm{Lip}_0 X$, $x_1 \in X$ and $y_1, y_2 \in Y$, $y_1 \neq y_2$ such that*

a) $\| f \|_X = 1$,

b) $f|_Y = 0$,

c) $f(x_1) = d(x_1, y_1) = d(x_1, y_2)$.

Proof. Let us suppose that there exists $f \in \mathrm{Lip}_0 X$, there exist $x_1 \in X$ and $y_1, y_2 \in Y$, $y_1 \neq y_2$ such that the conditions *a*), *b*), *c*) hold. Then

$$f(x_1) = d(x_1, y_1) = f(x_1) - f(y_1) = |f(x_1) - f(y_1)|,$$
$$f(x_1) = d(x_1, y_2) = f(x_1) - f(y_2) = |f(x_1) - f(y_2)|,$$

and by Theorem 4 in [1] it follows that the elements y_1, y_2 are two distinct elements of the best approximation for x_1. Consequently, Y is not semi-Chebyshevian.

If Y is not semi-Chebyshevian, then there exists an element $x_1 \in X$ and the elements $y_1, y_2 \in Y$, $y_1 \neq y_2$ such that

$$d(x_1, y_1) = d(x_1, y_2) = d(x_1, Y).$$

Then, the function $f(x) = d(x, Y)$, $x \in X$ is in $\mathrm{Lip}_0 X$ and verifies the condition $a)$, $b)$, $c)$. It follows that the asertion 2° is not fulfilled. The theorem is proved.

REFERENCES

[1] Mustăţa, C., *On the Best Approximation in Metric Spaces*, Mathematica — Revue d'Analyse Numérique et de la Théorie de l'Approximation, L'Analyse Numérique et la Théorie de l'Approximation, 4, 1, 45—50, (1975)

[2] Singer, I., *Cea mai bună aproximare în spaţii vectoriale normate prin elemente din subspaţii vectoriale, Anexa II*, Ed. Acad. R. S. România, Bucureşti, 1967.

Received, 9. II. 1978.

Universitatea „Babeş-Bolyai"
Cluj-Napoca
Institutul de Matematică