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ON THE NUMERICAL SOLUTION OF A NONLINEAR PARABOLIC EQUATION

by

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In a previous paper [5] we gave conditions under which the explicit difference analogue to (1.1)–(1.3) has a solution converging to the exact one. Some of conditions given there are very restrictive and exclude data interesting from the physical point of view. It is the aim of the present paper to show that the results hold under much larger and natural conditions.

1. Formulation of the problems. Consider the following problem:

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \text{ on } Q = \Omega \times]0, T[\tag{1.1}$$

$$u(x, 0) = u_0(x) \quad x \in \Omega \tag{1.2}$$

$$u(x, t) = u_1(x, t) \quad \text{on} \quad S = \Omega \times [0, T]. \tag{1.3}$$

As in [5] we suppose the following assumptions to hold:

(A)

(i) $u_0 \in C(\overline{\Omega}) \quad u_1 \in C(S), \quad u_0, u_1 \geqq 0$

(ii) $\lambda\varphi \in C^2(R_+)$, $\varphi(u)$ and $\varphi'(u) > 0$ for $u > 0$

$\varphi(0) = \varphi'(0) = 0, \quad \varphi''(u) \geqq 0$ for $u \geqq 0$.

We also suppose that $\Omega \subset R^2$ (for simplicity) and that this domain is bounded, regular and convex.

We called a function $u \in L^\infty(Q)$ a solution of (1.1)—(1.3) if

(i) $\frac{\partial \varphi}{\partial x} \in L^2(Q) \quad i = 1, 2$

(ii) Conditions (1.2)—(1.3) are fulfilled (in the generalized sense)

(iii) $u \geqq 0$ (a.e.) on Q

(iv) For any $f \in C^1(\bar{Q})$ such that $f|_{S_1} = 0$

$$\int_Q \left(u \frac{\partial f}{\partial t} - \sum_{i=1}^{2} \frac{\partial f}{\partial x_i} \frac{\partial \varphi(u)}{\partial x_i} \right) dxdt + \int_\Omega f(x, 0)\, u_0(x)\, dx = 0. \tag{1.4}$$

Here $S_1 = S \cup \{x, T) \mid x \in \bar{\Omega}\}$.

Remark. Condition (1.2) makes sense. Indeed, it will be shown that $\varphi(u)$ belongs to $L(0, T; H^1(\Omega))$. So by (1.1) it follows that $\frac{\partial u}{\partial t} \in L(0, T; H^{-1}(\Omega))$. This implies that $u \in C([0, T]; H^{-1}(\Omega))$, (see e.g. [4]. Lemma 1.2, Ch. I). We also note that u has well defined traces on S which depend continuously on t (e.g. [3] p. 70).

It was proved in [5] that the problem has at most one solution provided that $A(i)$ holds, $\varphi \in C(R_+)$ and $\varphi(u)$, $\varphi'(u) > 0$ for $u > 0$. In order to formulate the explicit numerical scheme associated to (1.1)—(1.3) we recall the notation from [5]:

$$R_h = \{(x, t) \in R^3 | x_i = k_i h,\ t = k_0 \tau,\ k_i = 0, \pm 1, \pm 2, \ldots, i = 1, 2;$$
$$k_0 = 0, 1, \ldots\}.$$

where $h, \tau > 0$ are the mesh-sizes,

$$\omega_{ij} = \{(x_1, x_2) | ih < x_1 < (i + 1)h,\ jh < x_2 < (j + 1)h\}$$

$$\Omega_h = \bigcup_{\omega \subset \Omega} \omega_{ij}, \qquad \Gamma_h = \partial\Omega_h,\ Q_h = \Omega_h \times]0, T].$$

We shall use the same notation for the mesh-points of R_h belonging to these domains, e.g. $Q_h = Q_h \cap R_h$, etc. For brevity we shall also set $U_{ij}(k)$ or even $U(k)$ instead of $U(ih, jh, k\tau)$. Now the difference problem is the following:

$$U_{\bar{t}}(k) = \Delta_h \varphi(U(k - 1)) \quad \text{on } Q_h \tag{1.5}$$

$$U(0) = u_{oh} \tag{1.6}$$

$$U(k)|_{\Gamma_h} = u_2(x, k\tau),\ x \in \Gamma_h,\ k = 0, 1, \ldots, K = \left[\frac{T}{\tau}\right]. \tag{1.7}$$

Here

$$U_{\bar{t}}(k) = \frac{U(k) - U(k - 1)}{\tau},$$

$$u_{oh} = u_0|\Omega_h,\quad u_2 : \Gamma_h \times \{k_0\tau,\ k_0 = 0, 1, \ldots, K\} \to R,$$

$$u_2(x, t) = u_1(x^*, t),\ x \in \Gamma_h,$$

where $x^* \in \partial\Omega$ is the nearest point to x (or one of them, if there are more, but always the same on all levels).

In the sequel we shall denote by M a positive constant such that $u_0, u_1 \leqq M$. Then, of course, $u_2 \leqq M$.

In order to compare the discrete and exact solutions we shall make use of the following extensions:

a) Constant extension. This attaches to each function U defined on the grid points of Q_h (or Ω_h) a step function $\tilde{U}$ defined on the domain $Q_h \subset Q$ (or Ω_h) by:

$$U(x, t) = U_{ij}(k) \quad (x, t) \in \omega_{ij} \times]k\tau, (k + 1)\tau[.$$

b) Multilinear extension. This assigns to the grid function U a continuous function U' defined by:

$$U' : \bar{Q}_h \to R$$

$$U'(x, t) = U_{ij}(k) + (U_{ij}(k))_{x_1}(x_1 - ih) + (U_{ij}(k))_{x_2}(x_2 - jh) +$$
$$+ (U_{ij}(k))_t(t - k\tau) + (U_{ij}(k))_{x_1x_2}(x_1 - ih)(x_2 - jh) +$$
$$+ (U_{ij}(k))_{x_1t}(x_1 - ih)(t - k\tau) + (U_{ij}(k))_{x_2t}(x_2 - jh)(t - k\tau),$$

$$(x, t) \in \omega_{ij} \times]k\tau, (k + 1)\tau[.$$

For a detailed study of these extensions we refer to [3].

2. Basic lemmas. In [5] we proved the following two lemmas:

Lemma 2.1. *Suppose that assumption* (A) *holds and that*

$$\lambda = 4 \frac{\tau}{h^2} \varphi'(M) \leqq 1.$$

Then the solution U *of* (1.5) — (1.7) *sarisfies the inequality*:

$$0 \leqq U \leqq M.$$

Lemma 2.2. *Suppose that the hypotheses of Lemma* 2.1 *are fulfilled and assume that*:

(i) $u_0 \in C^2(\Omega)$ *and* $\Delta u_0 \geqq 0$ *on* Ω;

(ii) u_1 *is nondecreasing in* t *on* $]0, T[$

Then

(j) $U_{\bar{i}}(k) \geq 0,\ k = 1, 2, \ldots, K$ on $\bar{\Omega}$

(jj) $\tau h^2 \sum_{\tilde{Q}_h} U_{\bar{i}}(k) \leqslant Mm(\Omega)$.

Here

$$\tilde{Q}_h = \bar{Q}_h \setminus \{(x,\ K\tau) \mid x \in \bar{\Omega}_h\}.$$

Remark. Condition $\Delta u_0 \geq 0$ can be replaced by $\Delta\varphi(u_0) \geq 0$. As we have already mentioned, conditions (i), (ii) are too restrictive, we shall therefore prove:

Lemma 2.3. *Suppose that*

(i) *Assumption* (A) *holds and* $u_0 \in C^2(\Omega)$;

(ii) $\lambda \leqslant 1$;

(iii) $\frac{\partial u_1}{\partial t}$ exists and is bounded on S;

(iv) $\varphi(\infty) = \infty$.

Then there exist constants h_0, $C > 0$ *such that*

$$\tau h^2 \sum_{\tilde{Q}_h} |U_{\bar{i}}(k)| \leqslant C, \text{ for } h \times h_0. \tag{2.1}$$

Proof: Let $\varepsilon > 0$ be a constant and $V_0 : \bar{\Omega} \to R$ a function such that

$$\Delta V_0 \geq |\Delta\varphi(u_0)| + \varepsilon \quad \text{on} \quad \Omega, \tag{2.2}$$

$$V_0|_\Gamma \geq \varphi(u_0)|_\Gamma,\ V_0 > 0,\ V_0 \in C^2(\bar{\Omega}).$$

We define v_0 by $v_0 = \varphi^{-1}(V_0)$ and $v_1 : S \to R$ as follows: $v_1 \geq 0$, $v_1 \in C(S)$, $\partial v_1/\partial t$ exists on S and

$$\frac{\partial v_1}{\partial t}(x,\ t) \geq c + \varepsilon,\ t \in]0,\ T[,\ v_1(x,\ 0) = v_0|_\Gamma$$

where $c \geq \max_S |\partial u_1/\partial t|$.

With the aid of v_0, v_1 as data functions we construct by means of the scheme (1.5)—(1.7) the discrete solution

$$V : Q_h \to R.$$

Thus taking into account the above Renark we get by Lemma 2.2 the following inequalities

$$V_{\bar{t}} \geq 0 \text{ on } Q_h$$

and

$$\tau h^2 \sum_{\tilde{Q}_h} V_{\bar{t}}(k) \geq M_1 m(\Omega)$$

For convenience we set $U_{\bar{i}}(k) = \bar{U}(k)$ and let us show that

$$\sum_{\Omega_h} |\bar{U}_{ij}(1)| \leqslant \sum_{\Omega_h} \bar{V}_{ij}(1). \tag{2.3}$$

Indeed by (2.2) for $h < h_0$, $\Delta_h\varphi(v_0) > |\Delta_h\varphi(u_0)|$. Hence according to (1.5) we get (2.3).

Suppose now that

$$\sum_{\Omega_h} |\bar{U}_{ij}(k-1)| \leqslant \sum_{\Omega_h} \bar{V}_{ij}(k-1) \tag{2.4}$$

and prove

$$\sum_{\Omega_h} |\bar{U}_{ij}(k)| \leqslant \sum_{\Omega_h} \bar{V}_{ij}(k). \tag{2.5}$$

From (1.5) we deduce that $(k \geq 2)$

$$U_{\bar{i}}(k) = U_{\bar{i}}(k-1) + \Delta_h(\varphi(k-1)) - \varphi(U(k-2)), \tag{2.6}$$

consequently

$$\bar{U}_{ij}(k) = (1 - 4\frac{\tau}{h^2}\tilde{\varphi}_{ij}(k-1))\bar{U}_{ij}(k+1) + \tag{2.7}$$

$$+ \frac{\tau}{h^2}[\tilde{\varphi}'_{i+1,j}(k-1)\bar{U}_{i+1,j}(k-1) + \tilde{\varphi}_{i-1,j}(k-1)\bar{U}_{i-1,j}(k-1) +$$

$$+ \tilde{\varphi}'_{i,j+1}(k-1)\bar{U}_{i,j+1}(k-1) + \tilde{\varphi}'_{i,j-1}(k-1)\bar{U}_{i,j-1}(k-1)]$$

and similarly for V. Here $\tilde{\varphi}'$ means an intermediary value between $\varphi'(U(k-1))$ and $\varphi'(U(k-2))$. Foe example:

$$\tilde{\varphi}'_{ij}(k-1) = \varphi'(U_{ij}(k-2) + \theta(U_{ij}(k-1) - U_{ij}(k-2))),$$

$$\theta \in]0,\ 1[.$$

From (2.7) we get $(k \geq 1)$,

$$\sum_{\Omega_h} |\bar{U}_{ij}(k)| \leqslant \sum_{\Omega_h} |\bar{U}_{ij}(k-1)| + \frac{\tau}{h^2}\sum_{\Gamma_h} \varphi'(\bar{U}_{ij}(k-1))|\bar{U}_{ij}(k-1)|$$

$$\sum_{\Omega_h} \bar{V}_{ij}(k) \leqslant \sum_{\Omega_h} \bar{V}_{ij}(k-1) + \frac{\tau}{h^2}\sum_{\Gamma_h} \varphi'(\bar{V}_{ij}(k-1))\bar{V}_{ij}(k-1).$$

Since $v_0|_\Gamma \geq u_0|_\Gamma$ and $\frac{\partial u_1}{\partial t} < \frac{\partial v_1}{\partial t}$ on S; h_0 can be taken so small that

$$u_1(x,\ t + \tau) \leqslant v_1(x,\ t) \text{ if } \tau \leqslant \tau_0 = \frac{\lambda h_0^2}{4\varphi(M)}.$$

in this case

$$\tilde U_{ij}(k-1) \leqslant \tilde V_{ij}(k-1) \text{ on } \Gamma_h$$

which implies

$$\varphi'(U_{ij}(k-1)) \leqslant \varphi'(\tilde V_{ij}(k-1)) \text{ on } \Gamma_h.$$

Here $\tilde U_{ij}(k-1) = U_{ij}(k-2) + \theta(U_{ij}(k-1) - U_{ij}(k-2))$, $\theta \in]0, 1[$ and similarly for V_{ij}.
On the other hand since $\partial v_1/\partial t > c$ on S it follows that

$$|\bar U_{ij}(k)| \leqslant \bar V_{ij}(k) \text{ on } \Gamma_h,\ k = 0, 1, \ldots, K.$$

Hence

$$\sum_{\Gamma_h} \varphi'(\tilde U_{ij}(k-1))|\bar U_{ij}(k-1)| \leqslant \sum_{\Gamma_h} \varphi'(\tilde V_{ij}(k-1))\bar V_{ij}(k-1)$$

so that (2.4) implies (2.5). Finally, if we observe that on the discrete boundary of Q

$$\sum |U_{\bar i}(k)| \leqslant \sum V_{\bar i}(k).$$

our lemma follows at once from Lemma 2.2.

C o r o l l a r y. *Under the hypotheses of the above lemma:*

$$\tau h^2 \sum_{\tilde\Omega_h} |\varphi(U(k)))_{\bar i}| < C, \tag{2.8}$$

C being a constant independent of h (and τ).

This is readily seen from

$$|\varphi(U(k))_{\bar i}| = \tilde\varphi'(k)|U_{\bar i}(k)| \leqslant \varphi'(M)|U_{\bar i}(k)^2|$$

and Lemma 2.2.

In order to estimate the differences in the space variables we give:

L e m m a 2.4. *Let* $U: \bar Q_h \to V$ *be the solution of* (1.5)—(1.7) *and suppose that conditions of Lemma* 2.3 *are fulfilled.*

Then there exist constants C, h_0, *independent of h* (and τ) *such that*

$$\tau h^2 \sum_{k=0}^{K-1} \sum_{\bar\Omega_h^*} (\varphi^2(U(k))_{x_1} + \varphi^2(U(k))_{x_2}) < C$$

for $h < h_0$.

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Here Ω_h^* *is a polygonal domain with sides parallel to* Ox_1, Ox_2; *convex in both directions and satisfying the conditions:*

(a) $\bar\Omega_h^* \subset \Omega^*$, $\bar\Omega^* \subset \Omega$

(b) *The sides of* $\partial\Omega_h^*$ *contain nodes of* R_h.

The proof becomes identical to that of [5] (Theorem 3.1), if instead of Lemma 2.2 we make use of Lemma 2.3.

L e m m a 2.5. ([1], Theorem 2.22) *Let* $1 \leqslant p < \infty$ *and let* $K \subset L^p(Q)$. *Suppose there exists a sequence* $\}Q_j\}$ *of subdomains of Q having the following properties:*

(i) *For each j,* $Q_i \subset Q_{j+1}$;

(ii) *For each j the set. of restrictions to* Q_j *of the functions in K is precompact in* $L^p(Q_j)$;

(iii) *For every* $\varepsilon > 0$, *there exists j such that*

$$\int_{Q \setminus Q_j} |u(x)|^p\, dx < \varepsilon \text{ for every } u \in K.$$

Then K is precompact in $L^p(Q)$.

3. Convergence of the numerical solution. Existence of the exact solution. The lemmas of the previous section show that the family of numerical solutions U (depending on h) is equibounded in in the discrete $W_1^1(Q_h)$ norm, i.e. the scheme (1.5)—1(.7) is stable in this norm (in fact we have shown that it is stable in in the norm of $W_1^1(0, T; W_2^1(\Omega_h))$). We are going now to discuss the the convergence of the discrete solution and prove that the limit is the exact solution we are looking for. We mention that, though our arguments refer to subsequences of approximate, solutions, they remain valid for the whole of the sequence, in view of the uniquenss of the exact solution.

According to the maximum principle given in Lemma 2.1, we have

$$0 \leqslant \tilde U \leqslant M,\quad 0 \leqslant U' \leqslant M \text{ on } Q,$$

if U is extended in $R_h \setminus \bar\Omega_h$ by any values which do not exceede M. Consequently, both families of functions are bounded in $L^q(Q)$, $1 \leqslant q \leqslant +\infty$. Hence there is a subsequence of steps $\{h_n\} \subset \{h\}$ so that U_{h_n} as well as U'_{h_n} converge weakly to $\chi \in L^2(Q)$, say. This limit is common as it was shown in [3] (see also [5], Lemma 4.3). In the sequel we shall write, for brevity, U_h insted of U_{h_n}.

Using the discrete variational formulation of problem (1.5)—(1.7), we have proved [5] the following theorems:

THEOREM 3.1. *Suppose that* $U : \overline{Q}_h \to R$ *is the solution of* (1.5)—(1.7) *and that the subsequence* U_h *is chosen as above. Assume that conditions of Lemma* 2.3 *are fulfilled. Then for* $h \to 0$:

(i) $\varphi(\tilde{U}_h) \to \varphi(\chi)$ in $L(Q')$, and a.e. on Q', $\forall \Omega' \subset \subset \Omega$

(ii) $\varphi(\tilde{U}_h) \rightharpoonup \varphi(\chi)$ in $L^2(Q')$, $\forall \Omega' \subset \subset \Omega$

(iii) $\varphi(\tilde{U}_h)_{x_i} \rightharpoonup \frac{\partial \varphi(x)}{\partial x_i}$, $i = 1, 2$, in $L^2(Q')$, $\Omega' \forall \subset \subset \Omega$.

Here $Q' = \Omega' \times]0, T[$, $\rightharpoonup$ designates the weak convergence, while $\subset \subset$ means the strict inclusion.

THEOREM 3.2. *Assume that conditions of the previous theorem hold. Then*

(i) $\lambda \in L^\infty(Q)$;

(ii) λ *satisfies* (1.4).

From Theorem 3.1 it follows that $\partial\varphi(\chi)/x_i \in L^2(Q)$, $i = 1, 2$. It is also clear that $\chi > 0$ on Q and that it satisfies condition (1.2), since $U_h \to \chi$ a.e. on Q'.

In order to show (ii) it suffices to write the difference problem (1.5)—(1.7) in the variational form (see [5], (5.1)]:

$$\int_Q \tilde{U}\tilde{\psi}_t - \tilde{\varphi}_{x_1}(U)\tilde{\psi}_{x_1} - \tilde{\varphi}_{x_2}(U)\tilde{\psi}_{x_2})dxdt +$$

$$+ \int_\Omega \tilde{u}_0(x)\tilde{\psi}(x, 0)dx = 0, \ \psi \in D([0, T[\times \Omega)$$

and pass to the limit.

THEOREM 3.3. *Suppose that* u *is the solution of problem* (1.1)—(1.3). *Then* $\varphi(\tilde{U})_h \to \varphi(u)$ *in* $L(Q)$. *The same is true for* U_h'.

Proof. Consider a sequence of subomains Q_j as in Lemma 2.5. Suppose $K = \{\varphi(\tilde{U}_h)\}_h$ (or $\varphi(U_h')$), then according to Lemmas 2.3 and 2.4. this is equibounded in $W_1^1(Q_j)$ and precompact in $L^1(Q_j)$ for any j. On the other hand by Lemma 2.1 K is bounded by the constant M independent of j on any point of Q. Thus all conditions of Lemma 2.5 are fulfilled, which completes the proof.

Remark. *Since the imbeddking*

$$W_1^1(Q) \to L^q(Q)$$

is compact, according to the Rellich-Kondrashov theorem, for any q *such that*

$$1 \leq q < 2$$

(in our particular case n = 2) *the convergence from the above theorem takes place in any* $L^q(Q)$, $1 \leq q < 2$.

REFERENCES

[1] Adams R. A., *Sobolev Spaces.* Academic Press, 1975.

[2] Descloux J., *On the Equation of Boussinesq.* Topics in Numerical Anaysis III, pp. 81—102, Academic Press, 1976.

[3] Ladyzhenskaia O. A., *Boundary-value Problems of the Mathematical Physics.* Nauka, Moscow, 1974 (in Russian).

[4] Lions J. L., *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, Paris 1969.

[5] Schechter E., *The Convergence of the Explicit Difference Approximation to Solve the Cauchy-Dirichlet Problem for Non-linear Parabolic Equations.* Mathematica, vol. 21 67—81 (1979).

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