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ON THE CLASSIFICATION OF DYNAMICAL SYSTEMS

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1. Introduction

The set of dynamical systems may be organized in many ways in categories according to the set of morphisms that one chooses (see [4]). Corresponding, one obtains many types of isomorphisms between dynamical systems, the most important being: the NS-isomorphism (isomorphism of Nemytskii-Stepanov type) and the GH-isomorphism (isomorphism of Gottschalk-Hedlund type). One considers [8] that it is impossible to find an effective method by which to decide if two dynamical systems are isomorphic, that is „the equivalence problem" is unsolvable. However, the problem was much studied yielding some partial answers. So is the result of L. MARKUS in [7] what contains a necessary and sufficient condition for two dynamical systems in the plane to be NS-isomorphic. His method of „separatrices" was generalized to dynamical systems defined on general metric spaces by himself in [9] and by N. P. BHATIA and L. M. FRANKLIN in [2], but with less success. Another approach to the problem was done by I. N. VRUBLEVSKAYA in [14] who uses a relation of equivalence among the trajectories of a dynamical system, relation defined by a kind of „regular deformation".

In this paper we propose a method similar to that of Vrublevskaya but based on a simpler definition of the deformation. For this we use the Pompeiu-Hausdorff metric in the case of the NS-isomorphism and a similar one in the case of the GH-isomorphism. This second metric we defined for the set of continuous functions between two metric spaces in [11] and have used then in a problem of dynamical systems in [12].

2. Basic notations and definitions

Let X be a metric space with a fixed metric $d = d_X$. For $a \in X$, $r > 0$ and $A \subset X$ we denote by:

(1) $$d(a, A) = \inf \; \{d(a, x) \; ; \; x \in A\}$$

the distance from a to A and by:

(2) $$V(a, r) = \{x \in X; \; d(a, x) < r\}$$

the ball of radius r and center a. For a function $f: X \to Y$ and a set $A \cup X$ we denote:

(3) $$f(A) = \{f(x) \; ; \; x \in A\}.$$

Definition 1. *A dynamical system on* X *is a continuous function* $\pi: X \times R \to X$ *that satisfies the following axioms:*

(i) $$\pi(x, 0) = x, \text{ for every } x \text{ in } X;$$

(ii) $$\pi(\pi(x, t), s) = \pi(x, t + s), \text{ for } x \text{ in } X, \; t \text{ and } s \text{ in } R.$$

Definition 2. *For any* $x \in X$ *one defines:*

a) *the motion* (through x) $\pi_x : R \to X$ by:

(4) $$\pi_x(t) = \pi(x, t) \; ;$$

b) *the trajectory of* x *by:*

(5) $$\gamma(x) = \{\pi(x, t) \; ; \; t \in R\} \; ;$$

c) *the positive limit set of* x *by:*

(6) $$L^+(x) = \{y \in X; \; \exists t_n \to +\infty, \; \pi(x, t_n) \to y\}.$$

Definition 3. *A point* $x \in X$ (and its trajectory) *is said to be:*
a) *critical, if* $\gamma(x) = \{x\}$;
b) *periodic if there is a* $p \neq 0$, *such that* $\pi(x, t + p) = \pi(x, t)$ *for all* $t \in R$;
c) *Lagrange stable, if* $\gamma(x)$ *is relatively compact*;
d) *positively Poisson stable, if* $x \in L^+(x)$;
e) ε*-stable* (in the sense of M. BERTOLINO [1]) *if any* ε*-neighbourhood of* $\gamma(x)$ *contains at least a trajectory distinct of* $\gamma(x)$.

Definition 4. *Let* π *and* σ *be dynamical systems on* X *respectively on* Y. *They are* NS-*isomorphic* (respectively GH-isomorphic) *if there exists a homeomorphism* $h: X \to Y$, *which preserves trajectories, that is:*

(7) $$\gamma(h(x)) = h(\gamma(x)), \text{ for all } x \in X$$

(respectively which makes commutative the following diagram:

(8) $$\begin{array}{ccc} X \times R & \xrightarrow{\pi} & X \\ \downarrow h \times 1_R & & \downarrow h \\ Y \times R & \xrightarrow{\sigma} & Y \end{array}$$

that is:

(8′) $$h(\pi(x, t)) = \sigma(h(x), t), \text{ for all } x \in X \text{ and } t \in R).$$

In what follows we shall use the set $I = [0, 1]$ and the function $\lambda: [0, \infty] \to I$ defined by:

(9) $$\lambda(t) = \begin{cases} \dfrac{t}{1+t} & \text{for } t \in [0, \infty) \\ 1 & \text{for } t = \infty \end{cases}$$

On the set of non-empty subsets of X we shall use the premetric P of Pompeiu-Hausdorff [6] defined for any $M, N \subset X$ by:

(10) $$P(M, N) = \lambda(\sup \; \{\sup \; \{d(x, N) \; ; \; x \in M\}, \; \sup \; \{d(y, M) \; ; \; y \in N\}\}).$$

Let $C(R,X)$ be the set of all continuous functions from R to X. We need several metrices for $C(R, X)$:

a) the uniform metric T, defined by:

(11) $$T(f, g) = \lambda \; (\sup \; \{d(f(t), g(t)) \; ; \; t \in R\}), \text{ for any } f, g \in C(R, X) \; ;$$

b) the metric K, which generates the compact — open topology, defined by:

(12) $$K(f, g) = \sum_{n=1}^{\infty} 2^{-n} \; \lambda(\max \; \{d(f(t), g(t)) \; ; \; |t| \leqslant n\}) \; ;$$

c) the metric S, of Pompeiu — Hausdorff type, which we defined in [11] by:

(13) $$S(f, g) = \lambda(\sup \; \{S_0(f, g), \; S_0(g, f)\}),$$

where

(13′) $$S_0(f, g) = \inf \{r > 0 \; ; \; \forall t \in R, \; \inf \{d(f(t), g(s)) \; ; \; |t - s| < r\} < r\},$$

with the usual convention: $\inf \emptyset = \infty$.

It is easy to check that for any $f, g \in C(R, X)$:

(14) $$K(f, g) \leqslant T(f, g) \text{ and } S(f, g) \leqslant T(f, g)$$

and, as we proved in [11], the identity function:

$$\mathrm{i}: (C(R,X),S) \to (C(R,X),K)$$

is continuous.

If it is necessary, we indicate by a lower index the space used in the definition of a certain metric (for exemple P_X).

3. Conditions for NS — isomorphism

Let X and Y be two metric spaces, and π and σ two dynamical systems on X respectively on Y. For theirs NS-isomorphism the two families of trajectories (of π and of σ) must correspond each to other by a homeomorphism $h: X \to Y$. But the families being infinite, it is difficult to check this correspondence. Thus it is natural to look for criterions which contains simpler conditions, at least necessary, for equivalence. We begin with some more general considerations.

On any family of subsets of X we consider the premetric P defined by (10). Let U be such a family.

D e f i n i t i o n 5. *We say that two sets* $M, N \subset X$ *are* U-equivalent (and denote by $M \sim N$ (rel U)) *if there exists a continuous function* h: $I \to U$ *such that* $h(0) = M$ *and* $h(1) = N$.

Remark 1. It is easy to verify that U-equivalence is an equivalence relation, hence it induces a partition of U. We denote the equivalence class of M by $\hat{M}$ and the quotient space by $\hat{U}$.

L e m m a 1. *If* U *and* W *are families of subsets of* X *respectively* Y, *then any continuous function* $F: U \to W$ *defines a continuous function* $\hat{F}: \hat{U} \to \hat{W}$ *by* $\hat{F}(\hat{M}) = \widehat{F(M)}$.

Proof. It is obvious that $M \sim N$ (rel U) implies $F(M) \sim F(N)$(rel W) so that $\hat{F}$ is well defined. Using the continuous canonical projections $i: U \to \hat{U}$ and $j: W \to \hat{W}$ we obtain the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{F} & W \\ \downarrow i & & \downarrow j \\ \hat{U} & \xrightarrow{\hat{F}} & \hat{W} \end{array}$$

From his commutativity results the continuity of $\hat{F}$ because $j \circ F$ is continuous (see [5]).

L e m m a 2. *If* $f: X \to Y$ *is a uniform continuous function with the property that* $f(M) \in W$ *for any* $M \in U$ *then the induced function* $f: U \to W$ *is continuous.*

Proof. We proceed by contradiction. Let us suppose that there exists a convergent sequence $M_n \to M$ such that $f(M_n) \nrightarrow f(M)$, that is one can find a $r > 0$ such that for any n_0 there exists $n > n_0$ for what $P_Y(f(M_n), f(M)) \geqslant r$. There are two possibilities:

a) $\sup\ \{d_Y(f(x), f(M))\ ;\ x \in M_{n_k}\} \geqslant r$;

b) $\sup\ \{d_Y(f(M_{n_k}), f(y))\ ;\ y \in M\} \geqslant r$,

for some $n_k \to \infty$. In the first case, for any k one can find a $x_{n_k} \in M_{n_k}$ such that:

$$d_Y(f(x_{n_k}), f(x)) \geqslant r, \quad \text{for any } x \in M. \tag{15}$$

But f being uniformly continuous there exists a $s > 0$, such that $d_X(x,y) < s$ implies $d_Y(f(x), f(y)) < r$, and because $M_{n_k} \to M$ one can find a k_0 with the property that for $k > k_0$, $P_X(M_{n_k}, M) < s$. For such a k, there exists $y_k \in M$ with $d_X(x_{n_k}, y_k) < s$, hence $d_Y(f(x_{n_k}), f(y_k)) < r$ which contradicts (15). In the case b) one fall on a similar contradiction using only the simple continuity of f.

C o n s e q u e n c e 1. *Let* X *and* Y *be compact metric spaces,* U *and* W *families of subsets of* X *respectively* Y. *If* $f(M) \in W$ *for any* $M \in U$ *and* $f^{-1}(N) \in U$ *for any* $N \in W$, *then the induced function* $\hat{f}: \hat{U} \to \hat{W}$ is a homeomorphism.

Let us apply these results to dynamical systems. For a dynamical system π on X we denote by $\Gamma\pi$ the family of all trajectories. How one knows [3], $\Gamma\pi$ is a partition of X and we consider on it, as before, the premetric P of Pompeiu — Hausdorff and the corresponding equivalence relation which we name in this case γ-equivalence. One obtains the quotient space $\hat{\Gamma}\pi$ and the following:

THEOREM 1. *If the dynamical systems* π *and* σ *defined on the compact metric spaces* X *respectively* Y *are* NS-*isomorphic, then the corresponding quotient spaces* $\hat{\Gamma}\pi$ *and* $\hat{\Gamma}\sigma$ *are homeomorphic.*

Remark 2. This condition is not sufficient for NS-isomorphism how shows the following:

Exemple 1. Let $X = Y = \{(x,y) \in R^2;\ x^2 + y^2 \leqslant 3\}$ as subspaces of R^2. The dynamical systems defined by autonomous differential equations (in polar coordinates):

$$(\pi) \qquad \begin{cases} \dot{\rho} = \rho(\rho - 1)(\rho - 3)\,; \\ \dot{\theta} = 1. \end{cases}$$

and

$$(\sigma)\quad \begin{cases} \dot{\rho} = \begin{cases} \rho(1-\rho) & \text{if } 0 \leqslant \rho < 1; \\ 0 & \text{if } 1 \leqslant \rho \leqslant 2; \\ (\rho-2)(\rho-3) & \text{if } 2 < \rho \leqslant 3; \end{cases} \\ \dot{\theta} = 1. \end{cases}$$

are not NS-isomorphic although the quotient spaces $\Gamma\pi$ and $\Gamma\sigma$ are homeomorphic.

As concerns the equivalence classes, we have the following:

THEOREM 2. *If the dynamical system π is defined on a complet, locally compact metric space X, then the class of γ-equivalence of a Lagrange stable trajectory contains only Lagrange stable trajectories.*

Proof. Let γ_0 be a Lagrange stable trajectory, $\gamma_1 \sim \gamma_0$ and $h: I \to \Gamma\pi$ continuous and such that $h(0) = \gamma_0$, $h(1) = \gamma_1$. Denote by $J = \{t \in I;\ h(t)$ is Lagrange stable$\}$. Of course $0 \in J$, that is $J \neq \emptyset$. Let $t_0 \in J$. The space X being locally compact and $\overline{h(t_0)}$ compact, $h(t_0)$ has a compact neighbourhood, that is J is open in I. But J is also closed in I. Indeed, if we suppose the contrary, there exists a sequence (t_n) in J which has a limit point $t_0 \notin J$. That is $\overline{h(t_0)}$ is not compact, i.e. it contains a sequence (x_n) which has no convergent subsequence. As $h(t_n) \to h(t_0)$ we have also $\overline{h(t_n)} \to \overline{h(t_0)}$ (in the topology induced by the premetric P on the set of subsets of X). If (ε_p) is a sequence of strictly positive numbers, then for every p, there is a natural number N_p such that $m > N_p$ implies $P(\overline{h(t_m)}, \overline{h(t_0)}) < \frac{\varepsilon_p}{3}$. Let $m_1 > N_1$ and $y_n^1 \in \overline{h(t_{m_1})}$ such that $d(y_n^1, x_n) < \frac{\varepsilon_1}{3}$ for any n. Because $\overline{h(t_{m_1})}$ is compact, there is a subsequence $(y_{n_k}^1)$ of (y_n^1) which is convergent. We may assume that $d(y_{n_k}^1, y_{n_l}^1) < \frac{\varepsilon_1}{3}$ for any k and l. Denoting x_{n_k} by x_k^1, we obtain the subsequence (x_k^1) of (x_n) such that $d(x_k^1, x_l^1) < \varepsilon_1$ for any k and l. Step by step, for $p = 2, 3, \ldots$ we obtain the sequence (x_n^p) such that $d(x_k^p, x_l^p) < \varepsilon_p$ for any k and l, (x_n^p) being subsequence of (x_n^{p-1}). So (x_n^n) is a Cauchy sequence and, X being complet, it is a convergent subsequence of (x_n) in contradiction with the assumption. After all, $J = I$, that is γ_1 is Lagrange stable.

Exemple 2. Let $X = R^2 - \{(0,0)\}$ with the usual Euclidean metric and π the dynamical system generated by the differential system:

$$\begin{cases} \dot{x} = x(x-1) \\ \dot{y} = 0 \end{cases}$$

Then the Lagrange stable trajectory $\{(x,1);\ 0 < x < 1\}$ is γ-equivalent with the trajectory $\{(x,0);\ 0 < x < 1\}$ which is not Lagrange stable, that is the theorem is not true if one renounces at completness.

L e m m a 3. *If the class of γ-equivalence of a critical point x is not a singleton, then x is ε-stable.*

The condition is not sufficient as shows the following:

Exemple 3. In the dynamical system generated by the differential system (in polar coordinates)

$$\begin{cases} \dot{\rho} = \begin{cases} \rho \sin \frac{1}{\rho}, & \text{for } \rho \neq 0; \\ 0 & , \text{for } \rho = 0; \end{cases} \\ \dot{\theta} = 1; \end{cases}$$

the origin is an ε-stable critical point whose class of γ-equivalence is a singleton.

4. Conditions for GH – isomorphism

For GH-isomorphism one can apply the theorem 1 because the GH-isomorphism of two dynamical systems implies theirs NS-isomorphism (the whole relation between them may be found in [13]). But we look for stronger criterions, specific to GH-isomorphism.

For a dynamical system π on X, let us consider the space of motions:

$$\Pi = \{\pi_x;\ x \in X\}$$

and the map $\pi^*: X \to \Pi$ defined by:

$$(16)\qquad \pi^*(x) = \pi_x$$

Taking on Π the compact-open topology, generated by the metric K, by the theorem of Fox [6], π^* is continuous. But attempting to introduce an equivalence relation on Π (with the metric K), as we did in $\Gamma\pi$, one obtain a triviality, namely the following result: a continuous map $h: I \to \Pi$ with the property that $h(0) = \pi_x$ and $h(1) = \pi_y$ exists iff there is a continuous map $f: I \to X$ such that $f(0) = x$ and $f(1) = y$. The use of the metric T for Π produces other difficulties. For exemple, two motions on the same trajectory may be not equivalent, as shows the following:

Exemple 4. Let π be a dynamical system on R defined by $\pi(x, t) = x \cdot \exp(t)$. If $x \neq y$ we have:

$$T(\pi_x, \pi_y) = \lambda(\sup \{|x - y| \exp(t);\ t \in R\}) = 1,$$

that is no motion is equivalent with another (taking the metric T for the space of motions).

In what follows we shall use only the metric S for Π, trying to avoid these difficulties. With this convention we give the following:

D e f i n i t i o n 6. *We say that two motions* π_x *and* π_y *are* Π*-equivalent* (and denote by $\pi_x \sim \pi_y$) *if there exists a continuous function* $h: I \to \Pi$, *such that* $h(0) = \pi_x$ *and* $h(1) = \pi_y$.

As above we denote the class of Π-equivalence of π_x by $\hat{\pi}_x$ and the quotient space $\Pi/_\sim$ by $\hat{\Pi}$.

L e m m a 4. *If* $\gamma(x) = \gamma(y)$, *then* $\pi_x \sim \pi_y$.

Proof. By the hypothesis $y = \pi(x, s)$ with some $s \in R$. Definig $h: I \to \Pi$ by

$$h(t) = \pi_{\pi(x,ts)}$$

it is continuous because

$$S(h(t_1), \; h(t_2)) \leqslant |t_1 - t_2| \cdot |s|$$

and $h(0) = \pi_x$, $h(1) = \pi_y$, that is $\pi_x \sim \pi_y$.

Remark 2. Thus S is more useful than the metric T. Also, generally it induces not trivial quotient spaces. For example, at the dynamical system appearing in exemple 4, one obtain three Π-equivalence classes (upon the sign of x).

Now, let us study some particular classes of motions. First of all we remark that the map $j: \Pi \to \Gamma\pi$ defined by $j(\pi) = \gamma(x)$, is continuous, because $P(\gamma(x), \gamma(y)) \leqslant S(\pi_x, \pi_y)$. Thus every class of Π-equivalence is „contained" in a class of γ-equivalence. So, for Π-equivalence hold properties analogous with those contained in the theorem 2 and in the lemma 3. Also we have:

L e m m a 5. *The set of motions positively Poisson stable is closed in* Π.

Proof. Let the motions π_{x_n} positively Poisson stable and π_x such that $S(\pi_{x_n}, \pi_x) \to 0$ for $n \to \infty$. Thus, for any natural p there is a n_p such that:

$$S(\pi_{x_{n_p}}, \pi_x) < \frac{2}{9p} \tag{17}$$

i.e. for some s_p, $|s_p| < 1/3p$ we have $d(x, \pi(x_{n_p}, s)) < 1/3\,p$. But, x_{n_p} being positively Poisson stable so is $\pi(x_{n_p}, s_p)$, thus one may find a $t_p > p + 1$, such that $d(\pi(x_{n_p}, s_p), \pi(x_{n_p}, s_p + t_p)) < 1/3p$. By (17) there is a τ_p such that $|s_p + t_p - \tau_p| < 1/3p$ and $d(\pi(x_{n_p}, s_p + t_p), \pi(x, \tau_p)) < 1/3p$. Finally, for any p we found a $\tau_p > p$ such that: $d(x, \pi(x, \tau_p)) < 1/p$ that is π_x is positively Poisson stable.

By the theorem of Poincaré-Bendixon [10] we have the following:

C o n s e q u e n c e 2. *For a dynamical system on the plane, the set of periodic motions is closed in* Π.

D e f i n i t i o n 7. *The function* $f: R \to X$ *has* ε*-period* τ *if for any,* $t \in R$, $d(f(t + \tau), f(t)) < \varepsilon$.

L e m m a 6. *If the class of* Π*-equivalence of a periodic motion* π_{x_0}, *with period* τ, *contains a motion which do not pass through* x_0, *then for every* $\varepsilon > 0$ *there is a motion, whose trajectory do not contain* x_0, *which has* ε*-period* τ *and such that* $S(\pi_x, \pi_{x_0}) < \varepsilon$.

Proof. Let $\varepsilon > 0$ fixed. Because π_{x_0} is uniformly continuous on $[-1, \tau + 1]$, there is a $\delta > 0$, $\delta < \min\left\{\frac{\varepsilon}{6}, \frac{1}{4}\right\}$, such that for any $t, s \in [-1, \tau + 1]$, with $|t - s| < 4\delta$, we have:

$$d(\pi(x_0, \; t), \; \pi(x_0, \; s)) < \frac{\varepsilon}{3}. \tag{18}$$

By hypothesis, there is a $x \in X$ such that $S(\pi_{x_0}, \pi_x) < \delta$ and $\pi_x(t) \neq x_0$ for any $t \in R$. Thus for any $t \in R$, there is a s_t, $|s_t - t| < 2\delta$ such that $d(\pi(x, t), \pi(x_0, s)) < 2\delta$. So we have succesively:

$$|s_{t+\tau} - (s_t + \tau)| \leqslant |s_{t+\tau} - (t + \tau)| + |s_t - t| < 4\delta$$

and, denoting $s_{t+\tau} = n\tau + t_0$, with $t_0 \in [0, \tau)$, $n \in \mathbf{Z}$:

$$|s_{t+\tau} - n\tau - (s_t + \tau - n\tau)| < 4\delta,$$

hence

$$s_{t+\tau} - n\tau, \; s_t + \tau(1 - n) \in [-1, \; \tau + 1]$$

and by (18):

$$d(\pi(x_0, s_{t+\tau}), \pi(x_0, s_t)) = d(\pi(x_0, s_{t+\tau} - n\tau), \pi(x_0, s_t + (1 - n)\tau)) < \varepsilon/3.$$

Finally:

$$d(\pi(x, t + \tau), \pi(x, t)) \leqslant d(\pi(x, t + \tau), \pi(x_0, s_{t+\tau})) + d(\pi(x_0, s_{t+\tau}), \pi(x_0, s_t)) +$$
$$+ \; d(\pi(x_0, \; s_t), \; \pi(x, t)) < 2\delta + \frac{\varepsilon}{3} + 2\delta < \varepsilon,$$

thus π_x satisfies all the expected conditions.

L e m m a 7. *If the map* $h: X \to Y$ *is uniformly continuous, then so is also the map* $h^*: C(R, X) \to C(R, Y)$ defined by:

$$(h^*(f))(t) = h(f(t)) \tag{19}$$

(using the metric S for the spaces of continuous functions).

Proof. For any $\varepsilon > 0$ there is a δ, $0 < \delta < \varepsilon$, with the property that $d_X(x, y) < \delta$ implies $d\,(h(x), \; h(y)) < \varepsilon$. If $f, g \in C(R, X)$ are such that $S_X(f, g) < \delta$, then for any $t \in R$ there is a $s_t \in R$ such that $|s_t - t| < \delta$ and $d_X(f(t), g(s_t)) < \delta$. Hence:

$$\min \{d_Y(h(f(t)), h(g(s))) ; |s - t| < \varepsilon\} \leqslant \min \{d_Y(h(f(t)),$$

$$h(g(s))) ; |s - t| < \delta\} \leqslant d_Y(h(f(t)), h(g(s_t))) < \varepsilon.$$

Changing the role of f and g we get $S_Y(h^*(f), h^*(g)) < \varepsilon$.

THEOREM 3. *Let π and σ be dynamical systems on compact metric spaces X respectively Y. If they are GH-isomorphic, then the quotient spaces $\hat{\Pi}$ and $\hat{\Sigma}$ of classes of equivalent motions are homeomorphic.*

Proof. We have the following diagram:

$$\begin{array}{ccccccc} X \times R & \xrightarrow{\pi} & X & \xrightarrow{\pi^*} & \Pi & \xrightarrow{i} & \hat{\Pi} \\ \downarrow{\scriptstyle h \times 1_R} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle \hat{h}} \\ Y \times R & \xrightarrow{\sigma} & Y & \xrightarrow{\sigma^*} & \Sigma & \xrightarrow{j} & \hat{\Sigma} \end{array}$$

where the first rectangle is from the GH-isomorphism of π and σ, π^* and σ^* are defined by (16), h^* by (19), i and j are canonical projections in quotient spaces and h is defined by:

$$\hat{h}(\hat{\pi}_x) = \hat{\sigma}_{h(x)}$$

From the continuity of h^*, one deduces (see [5]) the continuity of $\hat{h}$. To finish the proof it is enough to repeat the above considerations for h^{-1}.

Remark 3. We cannot use the commutativity of the last two rectangles and avoid so the proof of the continuity of h^*, because π^* and σ^* may be discontinuous.

Remark 4. As in the case of NS-isomorphism, the condition from theorem 3 is not sufficient for GH-isomorphism, how shows the following:

Exemple 5. Let π and σ be dynamical systems defined by the differential systems (in polar coordinates):

$$(\pi) \qquad \begin{cases} \dot{\rho} = 0 \\ \dot{\theta} = 1 \end{cases} \quad (\rho \leqslant 1)$$

$$(\sigma) \qquad \begin{cases} \dot{\rho} = 0 \\ \dot{\theta} = \rho \end{cases} \quad (\rho \leqslant 1)$$

The systems are NS-isomorphic but not GH-isomorphic, although the quotient spaces $\hat{\Pi}$ and $\hat{\Sigma}$ are homeomorphic being singletons.

Remark 5. The condition from theorem 3 is actually stronger than that of theorem 1, and a class of Π-equivalence may be properly „contained" in a class of γ-equivalence, as shows the following:

Exemple 6. Let σ be defined as above by:

$$(\sigma) \qquad \begin{cases} \dot{\rho} = 0 \\ \dot{\theta} = \theta\rho \end{cases}, \quad \rho \leqslant 1$$

One obtaine a single class of γ-equivalence but two motions on different trajectories are not Π-equivalent. Thus σ is NS-isomorphic with the dynamical system π from exemple 5 but is not GH-isomorphic with it.

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