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ON APPROXIMATE SOLVING BY SEQUENCES THE EQUATIONS IN BANACH SPACES

by

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Let X be a Banach space, Y a linear normed space, and the equation

(1) $$P(x) = \theta,$$

where $P: X \to Y$ is a continuous mapping, and θ is the null element of Y. We consider a sequence (x_n) of elements of X and a positive integer s.

DEFINITION 1. [4]. *The sequence (x_n) has the order s with respect to the mapping P, iff there exists a positive number α which does not depend on n so that*

$$||P(x_{n+1})|| \leqslant \alpha ||P(x_n)||^s \text{ for } n = 0, 1, 2, \ldots$$

DEFINITION 2. [4] *The sequence (x_n) is convergent of the order s with respect to the mapping P iff (x_n) has the order s with respect to the mapping P and it is convergent.*

In the present paper we study sufficient conditions on the mapping P and the sequence (x_n), for the convergence of the order s with respect to the mapping P of the sequence (x_n), so that the limit x^* of the sequence (x_n) is a solution of the equation (1).

Further on we shall suppose that $s \geqslant 2$.

THEOREM. *If the mapping* P, *the sequence (x_n) and the number $\delta > 0$ satisfy, in the sphere $S(x_0, \delta) = \{x : ||x - x_0|| \leqslant \delta\}$, the following conditions:*

(i) $$\sup \{||[u_1, u_2, \ldots, u_{s+1}; P]|| : u_1, u_2, \ldots, u_{s+1} \in S(x_0, \delta)\} \leqslant$$
$$\leqslant M < +\infty,$$

where $[u_1, u_2, \ldots, u_{s+1}; P]$ *is the divided difference of the s-th order of the mapping P in the different points* $u_1, u_2, \ldots, u_{s+1}$ [1], [2], [5];

(ii) *there exist the points* $x_{-s+1}, x_{-s+2}, \ldots, x_{-1} \in S(x_0, \delta)$ *and a positive numver A which does not depend on n so that*

$$||P(x_{n-s+1}) + \sum_{j=2}^{s} [x_{n-s+1}, x_{n-s+2}, \ldots, x_{n-s+j}; P] \times (x_{n+1} - x_{n-s+j-1})$$
$$(x_{n+1} - x_{n-s+j-2}) \ldots (x_{n+1} - x_{n-s+1})|| \leqslant ||P(x_n)||^s$$

for $n = 0, 1, 2, \ldots$ $(x_n \in S(x_0, \delta))$;

(iii) *there is a positive B which does not depend on n and k, so that*

$$||x_{n+1} - x_k|| \leqslant B||P(x_n)||$$

for $n = 0, 1, 2, \ldots$ *and* $k = -s+1, -s+2, \ldots, 0, 1, \ldots$ *with* $n \geqslant k$;

(iv) $h_0 = ||P(x_0)||(A + MB^s)^{\frac{1}{s-1}} = \eta_0 \cdot v < 1,$

$$\frac{B \cdot h_0}{v(1-h_0)} \leqslant \delta;$$

then

(j) *the sequence* (x_n) *is convergent of the order s with respect to the mapping P, and* $x^* = \lim_{n\to\infty} x_n$ *is a solution of the equation* (1), *i.e.* $P(x^*) = \theta$;

(jj) $x^* \in S(x_0, \delta)$;

(jjj) $||x_{n+1} - x_n|| \leqslant \frac{Bh_0^{s^n}}{v}$ *for* $n = 0, 1, \ldots$;

(jv) $||x^* - x_n|| \leqslant \frac{Bh_0^{s^n}}{v\left(1 - h_0^{s^n}\right)}$ *for* $n = 0, 1, \ldots$;

(v) $||P(x_n)|| \leqslant \frac{h_0^{s^n}}{v}$ *for* $n = 0, 1, \ldots$;

Proof. First we prove by induction the following relations

(2) $$x_i \in S, \quad i = 1, 2, \ldots;$$

(3) $$||x_i - x_{i-1}|| \leqslant \frac{B}{v} h_0^{s^{i-1}}, \quad i = 1, 2, \ldots;$$

(4) $$||P(x_i)|| \leqslant v^{s-1}||P(x_{i-1})||, \quad i = 1, 2, \ldots$$

a) For $n = k = 0$, (iii) gives

$$||x_1 - x_0|| \leqslant B||P(x_0)|| = \frac{Bv||P(x_0)||}{v} = \frac{Bh_0}{v} \leqslant \frac{Bh_0}{1-h_0} \leqslant \delta,$$

which means that (2) and (3) hold for $i = 1$.

By (ii) we have for $n = 0$.

$$||P(x_1)|| \leqslant \Big\| P(x_1) - [P(x_{-s+1}) +$$
$$+ \sum_{j=2}^{s} [x_{-s+1}, x_{-s+2}, \ldots, x_{-s+j}; P](x_1 - x_{-s+j-1})(x_1 - x_{-s+j+2}) \ldots (x_1 - x_{-s+1})]\Big\| +$$
$$+ \Big\| P(x_{-s+1}) + \sum_{j=2}^{s} [x_{-s+1}, x_{-s+2}, \ldots, x_{-s+k}; P](x_1 - x_{-s+j+1}) \ldots (x_1 - x_{-s+1})\Big\| \leqslant$$
$$\leqslant ||[x_{-s+1}, x_{-s+2}, \ldots, x_{-1}, x_0, x_1; P]|| + A||P(x_0)||^s \leqslant$$
$$\leqslant M \cdot B^s||P(x_0)||B^s + A||P(x_0)||^s = ||P(x_0)||^s(A + M \cdot B^s) =$$
$$= ||P(x_0)||^s\left[(A + MB^s)^{\frac{1}{s-1}}\right]^{s-1} = ||P(x_0)||^s v^{s-1},$$

i.e. for $i = 1$ (jv) is also true. Hence the relations (2) — (4) are verified for $i = 1$.

b) We suppose that the relations (2) — (4) are satisfied for a fixed $i > 1$.

c) We prove that the relations (2) — (4) are verified for $i + 1$.

Multiplying booth members of the inequality (4) by v, it results

(5) $$h_i = v||P(x_i)|| \leqslant v^s||P(x_{i-1})||^s = (v||P(x_{i-1})||)^s \leqslant (v^s||P(x_{i-2})||^s)^s =$$
$$= (v||P(x_{i-2})||)^{s^2} \leqslant \ldots (v||P(x_0)||)^{s^i} = h_0^{s^i}.$$

From (iii) we obtain for $n = k = i$

$$||x_{i+1} - x_i|| \leqslant B||P(x_i)|| = \frac{Bv||P(x_i)||}{v} = \frac{Bh_0^{s^i}}{v},$$

i.e. (3) is true for $i + 1$.

We have

$$||x_{i+1} - x_0|| \leqslant ||x_{i+1} - x_i|| + ||x_i - x_{i+1}|| + \ldots + ||x_1 - x_0|| \leqslant$$
$$\leqslant \frac{B}{v}\left(h_0^{s^i} + h_0^{s^{i-1}} + \ldots + h_0^s + h_0\right) \leqslant$$
$$\leqslant \frac{B}{v}(h_0 + h_0^2 + \ldots + h^i + \ldots) = \frac{Bh_0}{v(1-h_0)} = \delta$$

which shows that $x_{i+1} \in S(x_0, \delta)$.

Using (ii) for $n = i$, we can write

$$||P(x_{i+1})|| \leqslant \Big\| P(x_{i+1}) - \Big[P(x_{i-s+1}) +$$

$$+ \sum_{j=2}^{s} [x_{i-s+1}, x_{i-s+2}, \ldots, x_{i-s+j}; P](x_{i+1} - x_{i-s+j-1})(x_{i+1} - x_{i-s+j+2}) \ldots$$

$$\ldots (x_{i+1} - x_{i-s+1}) \Big] \Big\| + \Big\| P(x_{i-s+1}) +$$

$$+ \sum_{j=2}^{s} [x_{i-s+1}, x_{i-s+2}, \ldots, x_{i-s+j}; P](x_{i+1} - x_{i-s+j-1})(x_{i+1} - x_{i-s+j-2}) \ldots$$

$$\ldots (x_{i+1} - x_{i-s+1}) \Big\| \leqslant A||P(x_i)||^s + MB^s||P(x_i)||^s =$$

$$= ||P(x_i)||^s(A + MB^s) = v^{s-1}||P(x_i)||,$$

hence (4) is true for $i + 1$.

In conclusion the relations (2) – (4) hold for all the positive integers.

The property (4) shows that the sequence (x_n) has the order s with respect to the mapping P.

Now we prove that the sequence (x_n) is a Cauchy-sequence, hence it is convergent.

Using (3) we have

(6) $$||x_{n+p} - x_n|| \leqslant ||x_{n+p} - x_{n+p-1}|| + \ldots + ||x_{n+1} - x_n|| \leqslant$$

$$\leqslant \frac{B}{v}\left(h_0^{s^n} + h_0^{s^{n+1}} + \ldots + h_0^{s^{n+p}}\right) \leqslant$$

$$\leqslant \frac{B}{v} h_0^{s^n}(1 + h_0^s + h_0^{2s} + \ldots) = \frac{Bh_0^{s^n}}{v\left(1 - h_0^{s^n}\right)},$$

hence

$$\lim_{n\to\infty} ||x_{n+p} - x_n|| = 0$$

for every $p \in \mathbf{N}$, i.e. (x_n) is a Cauchy – sequence, and x being a Banach space the sequence (x_n) is convergent.

For $p \to \infty$ in the inequality (6) we obtain (jv). Also from (6) we obtain (jjj) puting $p = 1$.

From (2) it results that $x^* \in S(s_0, \delta)$.

For $i = n$ (5) implies

$$||P(x_n)|| \leqslant \frac{h_0^{s^n}}{v},$$

i.e. the relation (v) holds.

For proving the fact that x^* is a solution of the equation (1), in (v) we make $n \to \infty$, and so we obtain $(0 < h_0 < 1)$

$$\lim_{n\to\infty} ||P(x_n)|| = 0,$$

hence $\lim\limits_{n\to\infty} P(x_n) = P(x^*) = \theta$.

Remark. If P is Fréchet-differentiable, choosing the divided difference for which $[x, x; P] = P'(x)$, [3] we can reobtain the results of the paper [4].

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