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**L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION**
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# SOME IDENTITIES IN G-FUNCTIONS

by

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## 1. Introduction

The Barnes' $G$-function (see [2], p. 264), satisfies the recursion relation,

$$G(z+1) = \Gamma(z)G(z), \tag{1}$$

where $G(1) = 1$. The Eulerian integral of the first kind is defined by

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{2}$$

Then, by expressing $B(m, n)$ in terms of Barnes' $G$-functions and by making use of the identity

$$B(m, n) = B(m+1, n) + B(m, n+1), \tag{3}$$

BAGCHI [1] recently obtained an identity in $G$-functions. In this paper, we extend this work and obtain some more identities in $G$-functions.

## 2. Some identities in G-functions

In the sequel, we will make use of the following two results quite often. From (1) and (2), we have,

$$B(m, n) = \frac{G(m+1)}{G(m)} \cdot \frac{G(n+1)}{G(n)} \cdot \frac{G(m+n)}{G(m+n+1)} \tag{4}$$

and

$$(5)\qquad \binom{n}{r} = \frac{G(n+2)}{G(n+1)} \cdot \frac{G(r+1)}{G(r+2)} \cdot \frac{G(n-r+1)}{G(n-r+2)}.$$

**2.1.** It is known that

$$(6)\qquad \sum_k B(m, n+k) = B(m-1, n).$$

Hence, if we substitute for $B(m, n)$ from (4) in (6), and simplify, we get

$$(7)\qquad \sum_k \frac{G(n+k+1)}{G(n+k)} \; \frac{G(m+n+k)}{G(m+n+k+1)} = \frac{G^2(m)G(n+1)(Gm+n-1)}{G(m-1)G(m+1)G(n)G(m+n)}.$$

From the identities in $B(m, n)$,

$$(8)\qquad B(m, n+1) = \frac{n}{m} B(m+1, n)$$

and

$$(9)\qquad B(m, n+1) = \frac{n}{m+n} B(m, n)$$

we obtain the corresponding identities in $G$-functions:

$$(10)\qquad \frac{G(m+2)}{G(n+2)} = \frac{m}{n} \; \frac{G(n)\; G^2(m+1)}{G(m)\; G^2(n+1)}$$

and

$$(11)\qquad G(m+n+1) = \frac{n}{m+n} \; \frac{G(m+n)G^2(n+1)}{G(n+2)G(n)}$$

**2.2.** The hypergeometric function of the second kind for the unit value of the argument is

$$(12)\qquad F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$c \neq 0, -1, -2 \ldots ; c - a - b > 0 \ldots .$$

Now, using relation (1), we get the relation between Barnes' $G$-function and the hypergeometric function,

$$(13)\qquad F(a, b; c; 1) = \frac{G(c+1)\; G(c-a-b+1)}{G(c)\; G(c-a-b)} \cdot \frac{G(c-a)}{G(c-a+1)} \cdot \frac{G(c-b)}{G(c-b+1)}.$$

From Gauss' relations for contiguous hypergeometric functions, we can



get several relations for the Barnes' $G$-function by using (12). Here we consider only one such relation:

$$(14)\qquad (a-b)\; F(a, b; c; x) = aF(a+1, b; c; x) - bF(a, b+1; c; x).$$

If in (14) for $x = 1$, we substitute for $F(a, b; c; 1)$ from (13), we get

$$(15)\qquad G(c-a+b+1)\, G(c-a-b-1)\, [G(c-a)\;\; G(c-b)]^2 =$$

$$= G^2(c-a-b)\Big[\frac{a}{a-b} G(c-a-1)\;\; G(c-a+1)\;\; G^2(c-b) -$$

$$- \frac{b}{a-b} G(c-b-1)\;\; G(c-b+1)\; G^2(c-a)\Big], \quad a \neq b$$

Many more relations for Barnes' $G$-function can be obtained from the corresponding relations for contiguous hypergeometric functions.

In view of the importance of Barnes' $G$-function in the applied field we consider additional relations derived from the corresponding relations for binomial coefficients.

**2.3.** We know for the bimomial coefficient $\binom{n}{r}$,

$$(16)\qquad n\binom{n}{r} = r\binom{n+1}{r+1} + \binom{n}{r+1} =$$

$$(17)\qquad = (r+1)\binom{n}{r+1} + r\binom{n}{r},$$

$$(18)\qquad \binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

Substituting for $\binom{n}{r}$ from (5), we get the corresponding relations for the Barnes' $G$-functions

$$(19)\qquad G(n-r+2)\, G(n-r)\; [G(r+2)\; G(n+2)]^2 =$$

$$= [G(n-r+1)]^2\; [nG(r+1)G(r+3)G^2(n+2) -$$

$$- rG(n+3)G(n+1)G^2(r+2)],$$

$$(20)\qquad G(n-r+2)G(n-r)G^2(r+2) = \frac{n-r}{r+1} \cdot G^2(n-r+1)G(r+1)\; G(r+3),$$

and

$$(21)\qquad G(n-r+1)G(n-r+3)\,[G(r+1)G(n+2)]^2 =$$

$$= G^2(n-r+2)\,[G(n+3)G(n+1)\; G^2(r+1) - G(r)G(r+2)G^2(n+2)],$$

respectively.

From the relations,

$$\sum_{k=0}^{n}\binom{n}{k}=2^{n}, \text{ and}$$

(22)

$$\sum_{k=0}^{n}\binom{n}{k}^{2}=\binom{2n}{k},$$

we get the relations in $G$-functions,

$$\sum_{k=0}^{n}\frac{G(k+1)G(n-k+1)}{G(k+2)G(n-k+2)}=2^{n}\frac{G(n+1)}{G(n+2)} \tag{23}$$

and

$$\sum_{k=0}^{n}\left|\frac{G(k+1)G(n-k+1)}{G(k+2)G(n-k+2)}\right|^{2}=\frac{G^{4}(n+1)G(2n+2)}{G^{4}(n+2)G(2n+1)}, \tag{24}$$

respectively.

From the relations,

$$\sum_{k=0}^{p}\binom{p}{k}^{2}\binom{n+k}{2p}=\binom{n}{p}^{2} \tag{25}$$

and

$$\sum_{k=0}^{n}\binom{a+k}{k}=\binom{n+a+1}{n},$$

we get,

$$\sum_{k=0}^{p}\left[\frac{G(k+1)G(p-k+1)}{G(k+2)G(p-k+2)}\right]^{2}\frac{G(n+k+2)G(n+k-2p+1)}{G(n+k+1)G(n+k-2p+2)}= \tag{26}$$

$$=\left[\frac{G^{2}(p+1)}{G^{2}(p+2)}\frac{G(n+2)}{G(n+1)}\frac{G(n-m+1)}{G(n-m+2)}\right]^{2}\frac{G(2p+2)}{G(2p+1)}$$

and

$$\sum_{k=0}^{n}\frac{G(a+k+2)G(k+1)}{G(a+k+1)G(k+2)}=\frac{G^{2}(a+2)}{G(a+1)G(a+3)}\frac{G(n+1)}{G(n+2)}\frac{G(n+a+3)}{G(n+a+2)}, \tag{27}$$

respectively. In fact (27) can be written as:

$$\sum_{k=0}^{n}\frac{G(a+k+2)G(k+1)}{G(a+k+1)G(k+2)}=\frac{1}{(m+1)}\frac{G(n+1)G(n+a+3)}{G(n+2)G(n+a+2)}, \tag{28}$$



since

$$\frac{G(m)G(m+2)}{G^{2}(m+1)}=m.$$

Similarly, we can obtain several relations for Barnes' $G$-function from the corresponding relations for binomial coefficients. Here we list only two more such relations.

It is known that,

$$\sum_{k=0}^{m}\binom{2m+1}{2k+1}\binom{n+k}{2m}=\binom{2n}{2m} \tag{29}$$

and

$$\sum_{k=0}^{m}\binom{r}{k}\binom{n-r}{m-k}=\binom{n}{m}. \tag{30}$$

Then from (29) and (30), we get the following relations for $G$-functions,

$$\sum_{k=0}^{m}\frac{G(2k+2)G(2m-2k+1)G(n+k+2)\,G(n+k-2m+1)}{G(2k+3)\,G(2m-2k+2)\,G(n+k-1)G(n+k-2m+2)}= \tag{31}$$

$$=\frac{G(2n+2)G(2m+2)G(2n-2m+1)}{G(2n+3)G(2m+3)G(2n-2m+2)}$$

and

$$\sum_{k=0}^{m}\frac{G(k+1)G(r-k+1)G(m-k+1)G(n-r-m+k+1)}{G(k+2)G(r-k+2)G(m-k+2)G(n-r-m+k+2)}= \tag{32}$$

$$=\frac{G(r+1)G(m+1)G(n+2)G(n-r+1)G(n-m+1)}{G(r+2)G(m+2)G(n+1)G(n-r+2)G(n-m+2)},$$

respectively. Two more relations, similar to (31) and (32) can be obtained from:

$$\sum_{k=0}^{m}\binom{2m+1}{2k}\binom{n+k}{2m}=\binom{2n+1}{2m}$$

and

$$\sum_{k=0}^{n}\binom{k}{r}\binom{n-k}{m-r}=\binom{n+1}{m+1}.$$

## REFERENCES

[1] Bagchi, A. K., *A Short Note on Beta and Barnes G-function*. The Math Student, **37**, 214–215 (1969).
[2] Whittaker E. T., and Watson, G. N., *A Course of Modern Algebra*. Cambridge University Press, London, 1965.

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