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A FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS IN RANDOM NORMED SPACES

by

OLGA HADŽIĆ

(Novi Sad)

In this paper we shall prove a fixed point theorem for multivalued mappings in a complete random normed space $(S, \mathcal{F}, t)$ with continuous T-norm t. The notion of random normed space was introduced by ŞERSTNEV [5] and in [1] some fixed point theorems in random normed spaces are proved. First we shall give some definitions and theorems which we shall use in the later.

Let S be a real or complex linear space and Δ^+ be the set of all distribution functions F such that $F(0) = 0$. A *random normed space* is an ordered triple $(S, \mathcal{F}, t)$, where t is a T-norm stronger than T_m: $T_m(u. v) = \max\{u + v - 1, 0\}$ and $\mathcal{F}$ is a mapping of S into Δ^+ so that the following conditions are satisfied: (we shall denote $\mathcal{F}(p)$ by F_p)

1. $F_p = H \Leftrightarrow p = 0$ (0 is the neutral element in S)

2. $F_{\lambda p}(x) = F_p\left(\frac{x}{|\lambda|}\right)$ for every $p \in S$, $x \in \mathbf{R}$ and $\lambda \in K \setminus \{0\}$ where K is the scalar field.

3. $F_{p+q}(x + y) \geqslant t(F_p(x), F_q(y))$, for every $p, q \in S$ and every $x, y \in \mathbf{R}$.

The (ε, λ)-topology in $(S, \mathcal{F}, t)$ is introduced by the family of (ε, λ)-neighbourhoods of $v \in S$:

$$U_v(\varepsilon, \lambda) = \{u | u \in S,\ F_{u-v}(\varepsilon) > 1 - \lambda\},$$

where $\varepsilon > 0$ and $\lambda \in (0, 1)$ and if T-norm t is continuous then S is, in the (ε, λ)-topology, a Hausdorff linear topological space.

Every random normed space is a Menger space [4] if we take $F_{u,v} = F_{u-v}$, for every $u, v \in S$. In [4] the following theorem is proved.

THEOREM A. *Let* $(S, \mathcal{F}, t)$ *be a complete Menger space with continuous T-norm* t, A *be a closed subset of* S *and* $H: A \to A$ *such that:*

(1) $F_{Hp, Hq}(kx) \geqslant F_{p,q}(x)$, *for every* $x > 0$ *and every* $p, q \in S$,

where $k \in (0, 1)$. *If there exists* $p_0 \in A$ *such that:*

(2) $\sup_x G_{p_0}(x) = 1$, *where* $G_{p_0}(x) = \inf \{F_{p_n - p_0}(x) | n \in N\}$

and $p_n = Hp_{n-1}$ *for every* $n \in N$, *then there exists one and only one fixed point* p *of the mapping* H *and* $p = \lim_{n\to\infty} p_n$.

By $\mathcal{R}(M)$ we denote the family of all nonempty, closed and convex subsets of M, where M is a subset of a topological vector space. In [2] the following theorem is proved.

THEOREM B. *Let* E *be a Hausdorff topological vector space,* M *be a nonempty, convex and compact subset of* E, $\Phi: M \to \mathcal{R}(M)$ *be an upper semicontinuous mapping such that for every* $y \in M$ *the set:*

$$\Phi^{-1} y = \{x | y \in \Phi x\}$$

is open. Then there exists at least one fixed point of the mapping Φ.

Now, we shall prove a fixed point theorem for mapping $H + \Phi$ where H is a singlevalued and Φ is a multivalued mapping.

THEOREM. *Let* $(S, \mathcal{F}, t)$ *be a complete random normed space with continuous T-norm* t, M *be a nonempty, convex and compact subset of* S, H *be an affine mapping from* M *into* S *such that:*

(3) $F_{H(x_1)-H(x_2)}(k\varepsilon) \geqslant F_{x_1-x_2}(\varepsilon)$ *for every* $x_1, x_2 \in M$

and every $\varepsilon > 0$, *where* $k \in (0, 1]$ *and* $\Phi: M \to \mathcal{R}(S)$ *be an upper semicontinuous mapping such that* $HM + \Phi M \subseteq M$ *and the set* $\Phi^{-1} y$ *is open for every* $y \in S$. *Then there exists at least one fixed point of the mapping* $H + \Phi$.

Proof: First, we shall suppose that $k \in (0, 1)$. For every $y \in \overline{\Phi(M)}$ we shall define the mapping $G_y: M \to M$ in the following way:

$$G_y(x) = Hx + y, \text{ for every } x \in M.$$

Using the inequality (3) we conclude that:

$$F_{G_y(x_1)-G_y(x_2)}(k\varepsilon) \geqslant F_{x_1-x_2}(\varepsilon), \text{ for every } y \in \overline{\Phi(M)}$$

every $x_1, x_2 \in M$ and every $\varepsilon > 0$. Since T-norm t is continuous, S is a Hausdorff topological linear space and so it follows that the compact set M is bounded which means that for every neighbourhood V of zero

there exists $\delta > 0$ such that $M \subseteq \delta V$. If $V = \{x | F_x(\varepsilon) > 1 - \lambda\}$ we obtain that $\frac{x}{\delta} \in V$ i.e. that:

$$F_x(\varepsilon) > 1 - \lambda \text{ for every } x \in M$$

and so $F_x(\delta\varepsilon) > 1 - \lambda^\delta$ for every $x \in M$. From this it follows that the condition (2) is satisfied. Now, from Theorem A it follows that for every $y \in \overline{\Phi(M)}$ there exists one and only one fixed point Ry of the mapping G_y and so $Ry = HRy + y$ for every $y \in \overline{\Phi(M)}$. Let us prove that the mapping $R: y \to Ry$ is continuous. Suppose that $\lim_{n\to\infty} y_n = y$ $(y_n, y \in \overline{\Phi(M)})$ and let us show that $\lim_{n\to\infty} Ry_n = Ry$. Since $R: \overline{\Phi(M)} \to M$ and M is compact it follows that there exists a subsequence $\{y_{n(k)}\}_{k\in N}$ such that $\lim_{k\to\infty} Ry_{n(k)} = y^*$. Then from $Ry_{n(k)} = HR(y_{n(k)}) + y_{n(k)}$ we have:

$$y^* = Hy^* + y$$

and since the equation $z = Hz + y$ has one and only one solution Ry, we have $Ry = y^*$. Since every subsequence of the sequence $\{Ry_n\}$ has a convergent subsequence with the limit Ry we have that the mapping R is continuous. Further, we shall define the mapping $R^*: M \to 2^M$ in the following way: $R^*x = \bigcup_{y\in\Phi x} Ry$ for every $x \in M$. It is obvious that the mapping R^* is upper semicontinuous and we shall prove that R^*x is convex set, for every $x \in M$. Using the fact that mapping H is affine for every $\alpha, \beta \geqslant 0$ $\alpha + \beta = 1$ we have:

(4) $R(\alpha y_1 + \beta y_2) = \alpha R y_1 + \beta R y_2$ for every $y_1, y_2 \in \Phi x$.

So from the fact that Φx is convex, using (4) we conclude that R^*x is convex. It is obvious that there exists $R^{-1}: R\Phi(M) \to \Phi(M)$ and so:

$$(R^*)^{-1} y = \{x | y \in R^*x\} = \{x | R^{-1}y \in \Phi(M)\} = \Phi^{-1}(R^{-1}y)$$

Since $\Phi^{-n}y$ is open for every $y \in S$ we conclude that the set $(R^*)^{-1}$ is open and so all the conditions of Theorem B are satisfied for mapping R^* which implies the existence of an element $p \in M$ such that $p \in R^*p$ and so $p \in Hp + \Phi p$. Suppose now that $k = 1$. For every $n \in N$ we shall define the mappings H_n and Φ_n in the following way:

$$H_n x = \lambda_n H x, \text{ for every } x \in M;$$

$$\Phi_n x = \lambda_n \Phi x + (1 - \lambda_n) x_0, \text{ for every } x \in M;$$

where $\{\lambda_n\}_{n\in N}(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 1$.

Since $H_n M + \Phi_n M = \lambda_n(HM + \Phi M) + (1 - \lambda_n)x_0 \subseteq M$ and:

$$\Phi_n^{-1}y = \{x | y \in \Phi_n x\} = \{x | y \in \lambda_n \Phi x + (1 - \lambda_n)x_0\} =$$

$$= \left\{x | \frac{y - (1 - \lambda_n)x_0}{\lambda_n} \in \Phi x\right\} = \Phi^{-1}\left(\frac{y - (1 - \lambda_n)x_0}{\lambda_n}\right)$$

there exists, as we have proved, for every $n \in N$, $x_n \in M$ such that

$$x_n \in H_n x_n + \Phi_n x_n.$$

This means that there exists $y_n \in \Phi x_n$ such that $x_n = \lambda_n H x_n + \lambda_n y_n + (1 - \lambda_n)x_0$.

Then we have:

$$\lim_{n\to\infty} x_n - Hx_n - y_n = \lim_{n\to\infty} (\lambda_n - 1)Hx_n + (\lambda_n - 1)y_n + (1 - \lambda_n)x_0 = 0,$$

since $Hx_n + y_n \in M$ and M is bounded. Further, since M is compact, there exists a subsequence $\{n(k)\}_{k\in N} \subseteq N$ such that:

$$\lim_{k\to\infty} Hx_{n(k)} + y_{n(k)} = y^*$$

and so $\lim\limits_{k\to\infty} x_{n(k)} = y^*$. Since $y_{n(k)} \in \Phi x_{n(k)}$ and $\lim\limits_{k\to\infty} y_{n(k)} = y^* - Hy^*_y =$ we have $y^* - Hy^* \in \Phi y^*$ i.e. $y^* \in Hy^* + \Phi y^*$, which completes the proof.

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21000 NOVI SAD, YUGOSLAVIA
Faculty of Sciences, Department of Mathematics, Dr. Ilije Djuričića 4

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