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BOUNDARIES OF SMOOTH STRICTLY CONVEX PLANE SETS

by

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In a previous note [5] we proved the following characterization of boundaries of smooth strictly convex plane sets:

THEOREM 1. *A plane compact set S is the boundary of a smooth strictly convex set if and only if the following conditions hold:*

(*a*) $$\forall x, y, z \in S \quad S \cap \text{int conv } \{x, y, z\} = \emptyset.$$

(*b*) *For every triangle $P_1P_2P_3$ in $\mathbf{R}^2$ there is one and only one triangle $P_1'P_2'P_3'$ with sides parallel to the sides of $P_1P_2P_3$ and of the same orientation as $P_1P_2P_3$ and which is inscribed in S in the sense that $P_1', P_2', P_3' \in S$.*

In this paper we shall prove another theorem, which characterizes the boundaries of smooth strictly convex plane sets. We shall maintain condition (b) of Theorem 1, but instead of condition (a) we shall use a kind of "smoothness" condition, namely condition (i) of Theorem 3. Conditions of this nature have been studied also by W.R. ABEL and L.M. BLUMENTHAL [1]. The terminology will be the same as in [5]. We shall denote by int S, bd S, card S and conv S the interior, the boundary, the cardinal and respectively the convex hull of the set S.

In the sequel we need also the following theorem of A. MARCHAUD [7]:

THEOREM 2. *In order that a continuum C in an n-dimensional Euclidean space be a Jordan curve without double points it is necessary and sufficient that for any $x \in C$ and for every sufficiently small open connected set G containing x we have*

$$\text{card } \{C \cap \text{bd } G\} \leq 2.$$

We can now enounce our main result:

THEOREM 3. *In order that a plane compact set S be the boundary of a smooth strictly convex set it is necessary and sufficient that the following two conditions are verified:*
(i) For every $p \in S$ *and for every* $\varepsilon > 0$ *there exists a positive number* $\delta(p, \varepsilon)$ *such that for every triplet r, s, t of points of* $S \cap D(p, \delta(p, \varepsilon))$, $r \neq s \neq t \neq r$ *we have* $\max \sphericalangle (r, s, t) > \pi - \varepsilon$, *where* $D(p, \delta(p, \varepsilon))$ *is the interior of the disc with center p and radius* $\delta(p, \varepsilon)$.
(ii) For every triangle abc in $\mathbf{R}^2$ *there is one and only one triangle a'b'c' with sides parallel to the sides of abc, of the same orientation as abc and which is inscribed in S in the sense that* $a', b', c' \in S$.

By $\max \sphericalangle (r, s, t)$ we denote the maximum angle of the triangle *rst*.
Proof of the sufficiency. Let S be a plane compact set which verifies conditions (*i*) and (*ii*). Let $a(0)$, $b(0)$, $c(0)$ and $c(1)$ be four points of S such that $a(0)b(0)c(0)c(1)$ is a convex quadrilateral. For a given $t \in [0, 1]$ consider the point $c_t = tc(1) + (1-t)c(0)$. By condition (*ii*) there corresponds to the triangle $a(0)b(0)c_t$ one and only one triangle $a(t)b(t)c(t)$ with sides parallel to those of the traingle $a(0)b(0)c_t$ of the same orientation as the triangle $a(0)b(0)c_t$ and which is inscribed in S, i.e. for which $a(t), b(t), c(t) \in S$. With $a(t)$, $b(t)$ and $c(t)$ defined in this way we can consider the following three mappings:

$$a: [0, 1] \to S,$$
$$b: [0, 1] \to S,$$
$$c: [0, 1] \to S.$$

Lemma 1. *The three mappings* $a: [0, 1] \to S$, $b: [0, 1] \to S$ *and* $c: [0, 1] \to S$ *are continuous. We have in plus* $a(0) = a(1)$ *and* $b(0) = b(1)$ *and the points* $c(0)$ *and* $c(1)$ *are connected by a way in S.*

We say that two points *a* and *b* are connected by a way in S if there is a continuous mapping $f: [0, 1] \to S$ such that $f(0) = a$ and $f(1) = b$.
Proof of Lemma 1. Suppose for instance that the mapping $c: [0, 1] \to S$ is not continuous. There is then a sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \in [0, 1]$ and $\lim_{n\to\infty} t_n = t^0$ such that the sequence $\{c(t_n)\}_{n=1}^{\infty}$ doesn't converge to $c(t^0)$. Then there exists and $\varepsilon_0 > 0$ such that in the exterior of the open disc $D^0 = D(c(t^0), \varepsilon_0)$ of centre $c(t^0)$ and of radius ε_0 there are an infinity of points $c(t_n)$. Since $S - D^0$ is a compact set, we can extract a subsequence $\{n_k\}_{k=1}^{\infty}$ for which

$$\lim_{k\to\infty} c(t_{n_k}) = c^* \in S - D^0$$

$$\lim_{k\to\infty} a(t_{n_k}) = a^* \in S \tag{1}$$

$$\lim_{k\to\infty} b(t_{n_k}) = b^* \in S.$$

We consider now the following three cases:
I) $a^* = b^* = c^*$.
II) Only two of the points a^*, b^* and c^* coincide.
III) a^*, b^* and c^* are three distinct points.
I) In the first case denote with $m = \max_{0\le t\le 1} \max \sphericalangle (a(0), b(0), c_t)$. By the convexity of the quadrilateral $a(0)b(0)c(0)c(1)$ follows immediately $m < \pi$. Let be $\varepsilon = \frac{\pi - m}{2} > 0$. By condition (*i*) there corresponds to a^* and ε a number $\delta(a^*, \varepsilon)$ such that for every triplet of points $r, s, t \in S \cap D(a^*, \delta(a^*, \varepsilon))$, $r \neq s \neq t \neq r$ we have $\max \sphericalangle (r, s, t) > \pi - \varepsilon$. It follows from (1) that there is a number N such that for $k > N$ we have $a(t_{n_k})$, $b(t_{n_k})$, $c(t_{n_k}) \in S \cap D(a^*, \delta(a^*, \varepsilon))$. Therefore we have for $k > N$ the inequality $\max \sphericalangle (a(t_{n_k}), b(t_{n_k}), c(t_{n_k})) > \pi - \varepsilon$. But as the triangles $a(t_{n_k})b(t_{n_k})c(t_{n_k})$ and $a(0)b(0)c_{t_{n_k}}$ have parallel sides, we have also $\max \sphericalangle (a(0), b(0), c_{t_{n_k}}) > \pi - \varepsilon = \pi - \frac{\pi - m}{2} = \frac{\pi + m}{2} > m$, which contradicts the definition of *m*.
II) Suppose now that we are in the second case, i.e. only two of the points a^*, b^*, c^* coincide. If we denote with $\min \sphericalangle (a, b, c) = \min \{\sphericalangle abc, \sphericalangle bca, \sphericalangle cab\}$ we obtain in this case

$$\lim_{k\to\infty} \min \sphericalangle (a(t_{n_k}), b(t_{n_k}), c(t_{n_k})) = 0$$

Since the corresponding sides of the triangles $a(t_{n_k})b(t_{n_k})c(t_{n_k})$ and $a(0)b(0)c_{t_{n_k}}$ are parallel, it follows also that

$$\lim_{k\to\infty} \min \sphericalangle (a(0), b(0), c_{t_{n_k}}) = 0,$$

which contradicts our hypothesis that $a(0)b(0)c(0)c(1)$ is a convex quadrilateral.
III) Consider now the third case, when the three points a^*, b^*, c^* are distinct. Since the triangles $a(0)b(0)c_{t_{n_k}}$ and $a(t_{n_k})b(t_{n_k})c(t_{n_k})$ have the corresponding sides parallel and are of the same orientation, the same holds for the limit positions $a^*b^*c^*$ and $a(0)b(0)c_{t^0}$. But the triangle $a(t^0)b(t^0)c(t^0)$ is also inscribed in S, has its sides parallel to those of the triangle $a(0)b(0)c_{t^0}$ and has the same orientation as the triangle $a(0)b(0)c_{t^0}$. Because $c^* \in S - D^0$, $c(t^0) \in D^0$ we have $c^* \neq c(t^0)$, which contradicts the unicity part of condition (ii).
It follows that $c: [0, 1] \to S$ is a continuous mapping and thereby we can say that the points $c(0)$ and $c(1)$ are connected in S by a way.
The proof of the continuity of the mappings $a: [0, 1] \to S$ with $a(0) = a(1)$ and $b: [0, 1] \to S$ with $b(0) = b(1)$ is the same as that of the continuity of the mapping $c: [0, 1] \to S$.

L e m m a 2. *Let S be a plane compact set which verifies the conditions (i) and (ii) and let a and b be two arbitrary points of S, $a \neq b$. There exists then a parallel to the line $L(a, b)$ that meets S in at least two points.*

Proof of Lemma 2. Let $L(a, b)$ be the line determined by the points a and b. Let us suppose that on every parallel to $L(a, b)$ different from this, there is at most one point of S. We can choose a rectangular system of coordinates in $\mathbf{R}^2$ such that the Ox-axis coincides with $L(a, b)$. We can also suppose that a and b are the points of $S \cap L(a, b)$ of minimal and respectively of maximal abscissa. Let c_0 and d_0 be the two points in $\mathbf{R}^2$ of positive and respectively of negative ordinate such that the triangles abc_0 and abd_0 are equilateral. Let a_1b_1c be the triangle which corresponds by (ii) to the triangle abc_0 and let a_2b_2d be the triangle which corresponds by (ii) to the triangle abd_0. By our supposition that for every line L' parallel to $L(a, b)$ with $L' \neq L(a, b)$ we have card $L' \cap S \leq 1$, it follows that $a_1, b_1, a_2, b_2 \in L(a, b)$, $c \in S$ has a positive ordinate and $d \in S$ has a negative ordinate. It results then that $abcd$ is a convex quadrilateral inscribed in S. Applying Lemma 1 we see that a and c are connected in S by a way α and the points b and c are connected in S by a way β. By our supposition that every parallel to the Ox-axis cuts S in at most one point it follows that the segment $[a, b] \subset S$ and there follows also the existence of a point e of the segment $[a, b]$ such that e is connected with c by a way γ in S, all points of γ with the exception of e having positive ordinates.

We have now to distinguish two cases:

1) $\gamma \cap \text{int conv } \{a, b, c\} \neq \emptyset$

2) $\gamma \cap \text{int conv } \{a, b, c\} = \emptyset$.

In the first case consider a point c' of $\gamma \cap \text{int conv } \{a, b, c\}$. Denote by a' the intersection point of the segment $[a, b]$ with the parallel to $L(a, c)$ through c' and with b' the intersection point of the segment $[a, b]$ with the parallel to $L(c, b)$ through c'. Both triangles abc and $a'b'c'$ are then inscribed in S, they have the corresponding sides parallel and they are of the same orientation. But this contradicts the unicity part of the condition (ii).

In the second case we can suppose $e = a$. If we choose now a point f interior to the triangle abc, it follows that there doesn't exist any triangle inscribed in S of the same orientation as the triangle abf and with sides parallel to the corresponding sides of the triangle abf. We got again a contradiction, this time to the existence part of condition (ii). This completes the proof of Lemma 2.

L e m m a 3. *If S is a plane compact set for which conditions (i) and (ii) are verified, then S is a continuum.*

Proof of Lemma 3. Let a and b be two arbitrary points of S. According to Lemma 2 there exist two points c and d of S such that $abcd$ is a trapezium. Applying now Lemma 1 it results that the points a and b are connected in S by a way. It follows that S is connected. Since S was supposed to be compact, S is a continuum.

For each $p \in S$ and for any $\varepsilon > 0$ there exists accordingly to condition (i) of Theorem 3 a number $\delta(p, \varepsilon)$ such that for any triplet r, s, t of points from $S \cap D(p, \delta(p, \varepsilon))$ with $r \neq s \neq t \neq r$ we have max $\measuredangle(r,s,t) > \pi - \varepsilon$. We have then

L e m m a 4. *Let S be a set of the Euclidean space $\mathbf{R}^2$ for which condition (i) holds. For every $p \in S$ and $\delta < \delta(p, \pi/3)$ we have then*

$$\text{card } \{S \cap C(p, \delta)\} \leq 2,$$

where $C(p, \delta)$ is the circle with center p and of radius δ.

Proof of Lemma 4. Let us suppose the contrary, i.e. the existence of a point $p \in S$ and of a $\delta_0 < \delta(p, \pi/3)$ such that we have card $\{S \cap C(p, \delta_0)\} \geq 3$. Let then r, s, t, be a triplet of distinct points of the set $S \cap C(p, \delta_0)$. At least one of the arcs rs, st or tr has then a length $\leq 2\pi\delta_0/3$. We can suppose without loss of generality that the arc rs has a lenght $\leq 2\pi\delta_0/3$. The triplet of points r, p, s verifies then $p \neq r \neq s \neq p$ and $r, p, s \in S \cap D(p, \delta(p, \pi/3))$. But as $\pi \geq \measuredangle prs + \measuredangle psr = \pi - \measuredangle rps \geq \pi/3$ it follows for the isosceles triangle rps (with $rp = sp$) that $\pi/6 \leq \measuredangle prs = \measuredangle psr \leq \pi/2$. Hence we have max $\measuredangle(r, p, s) \leq 2\pi/3$ and we got a contradiction to property (i).

Applying now Lemma 4 to a plane compact set S, which verifies conditions (i) and (ii) of Theorem 3, we get for $\delta < \delta(p, \pi/3)$ the inequality card $\{S \cap C(p, \delta)\} \leq 2$, where $C(p, \delta)$ is the circle of center p and radius δ. By Theorem 2 of A. MARCHAUD it follows that S is a Jordan curve without double points.

Our objective is now to prove that S is a closed Jordan curve. Suppose that S is an open Jordan curve of endpoints $c(0)$ and $c(1)$, $c(0) \neq c(1)$. Since S verifies condition (ii) it results immediately that S must have points beyond the line $L(c(0), c(1))$ determined by the points $c(0)$ and $c(1)$. Let e be a point of S having a maximal distance to the line $L(c(0), c(1))$. Such a point e exists because S is a compact set. Every line parallel to $L(c(0), c(1))$ between e and $L(c(0), c(1))$ cuts S in at least two points. Let L be such a parallel line to $L(c(0), c(1))$ and let $a(0)$ and $b(0)$ be two points of $S \cap L$ such chosen that one of them belongs to the Jordan curve with endpoints $c(0)$ and e, while the other belongs to the Jordan curve with endpoints e and $c(1)$ and $a(0)b(0)c(0)c(1)$ is a trapezium. With $a(0)$, $b(0)$, $c(0)$ and $c(1)$ such determined, consider now the mapping $c: [0, 1] \to S$ defined in Lemma 1. This mapping is continuous by Lemma 1. But since a continuous mapping applies a connected set into a connected set, it follows that there is a $t \in [0, 1]$ such that $c(t) = e$. This means that there is a triangle $a(t)b(t)c(t)$ inscribed in S with sides parallel to the corresponding sides of the triangle $a(0)b(0)c_t$ (where $c_t = tc(1) + (1 - t)c(0))$ and of the same orientation as the triangle $a(0)b(0)c_t$ and in plus we have $c(t) = e$. But then the points $a(t)$ and $b(t)$ of S would have a distance to $L(c(0), c(1))$ greater than the distance of

e to the line $L(c(0), c(1))$. This contradicts the definition of the point e. It follows that S is a closed Jordan curve without double points.

We shall now prove that S is a convex Jordan curve, i.e. S has to coincide with the boundary of its convex hull, $S = \text{bd conv } S$. Let us suppose $S \neq \text{bd conv } S$. Since we have $S \subset \text{conv } S = \text{bd conv } S \cup \cup \text{int conv } S$ and S is a closed Jordan curve without double points it follows that $S \cap \text{int conv } S \neq \emptyset$ and there has to be a point $x \in \in \text{bd conv } S$ such that $x \notin S$. By the theorem of W. FENCHEL (see [3] or [4]) on the convex hull of a connected set, there are two points $c(0)$ and $c(1)$ in S such that $x \in \text{conv } \{c(0), c(1)\}$ i.e. x belongs to the line segment $[c(0), c(1)]$. Since S is a compact set, we can choose the points $c(0)$ and $c(1)$ in S such that $]c(0), c(1)[\cap S = \emptyset$. $L(c(0), c(1))$ is a supporting line for conv S. Let L' be the other supporting line for conv S which is parallel to $L(c(0), c(1))$. Since S is a compact set we have $L' \cap S \neq \neq \emptyset$. Let e be a point of $L' \cap S$. The points $c(0)$ and $c(1)$ determine on the closed Jordan curve two simple arcs, namely the arc $c(0)ec(1)$ and an arc $c(0)yc(1)$ which have only their endpoints in common. The arc $c(0)yc(1)$ with the exception of its endpoints $c(0)$ and $c(1)$ is contained in int conv S. Let f be a point on the arc $c(0)yc(1)$ at a maximal distance from the line $L(c(0), c(1))$. Let $b(0)$ be a point on the subarc $c(0)f$ and $a(0)$ be a point on the subarc $c(1)f$ such that the line $L(a(0), b(0))$ is parallel to the line $L(c(0), c(1))$. The quadrilateral $a(0)b(0)c(0)c(1)$ is then a trapezium and we can again define the mappings $a: [0, 1] \to S$, $b: [0, 1] \to \to S$ and $c: [0, 1] \to S$ as those in Lemma 1, i.e. such that $a(t)b(t)c(t)$ is the unique triangle inscribed in S with sides paralel to those of the triangle $a(0)b(0)c_t$ and of the same orientation as the triangle $a(0)b(0)c_t$, where $c_t = tc(1) + (1-t)c(0)$. Since this three mappings are continuous accordingly to Lemma 1, it follows that the image of the interval $I = = [0, 1]$ under the mapping $c: [0, 1] \to S$ is a connected subset of S, which contains the points $c(0)$ and $c(1)$. It follows that $c(I)$ must contain either the point e or the point f.

The point e cannot be contained in $c(I)$, because there doesn't exist any $t \in [0, 1]$ such that $c(t) = e$ and that the triangle $a(t)b(t)c(t)$ is inscribed in S, has parallel sides to those of the triangle $a(0)b(0)c_t$ and is of the same orientation as the triangle $a(0)b(0)c_t$, where $c_t = tc(1) + + (1-t)c(0)$. For each $t \in [0, 1]$ follows $b(t) \neq c(0)$, $b(t) \neq c(1)$, $a(t) \neq \neq c(0)$ and $a(t) \neq c(1)$. For, if we suppose the contrary, it results that $c(t)$ and f are on opposite sides relative to the line $L(c(0), c(1))$, which is impossible since $L(c(0), c(1))$ is a supporting line for conv S. Since $b(0)$ belongs to the arc $c(0)fc(1)$ and $b: [0, 1] \to S$ is a continuous mapping it follows that $b(t)$ belongs to the arc $c(0)fc(1)$ for every $t \in [0, 1]$. If we suppose that for some $t \in [0, 1]$ we have $c(t) = f$, it results that $a(t)$ and $b(t)$ have a greater distance to the line $L(c(0), c(1))$ than the point f, which contradicts our choice of the point f. Hence we have proved that $S = \text{bd conv } S$.

The proof of the strict-convexity of conv S is the same as in the proof of Theorem 3 in [5].

We have further to prove that conv S is a smooth set. Let us suppose the contrary i.e. the existence of a point $a \in S = \text{bd conv } S$ at which we have two supporting lines D_1 and D_2, which form an angle α with vertex a. We consider on the bisector B of the angle α a sequence of points $\{x_n\}_{n=1}^{\infty}$ which tends monotone on B to the vertex a. The perpendicular in x_n on the bisector B will meet S in at least two points. Let b_n and c_n be two of these points. It is obviously that with $\lim_{n\to\infty} x_n = a$ we have also $\lim_{n\to\infty} b_n = a$ and $\lim_{n\to\infty} c_n = a$. We set now $a_n = a$, $n = 1, 2, \ldots$ The following inequalities are then obviously:

$$\measuredangle a_n b_n c_n \geqq \frac{\pi - \alpha}{2}, \quad \measuredangle a_n c_n b_n \geqq \frac{\pi - \alpha}{2} \text{ and } \measuredangle b_n a_n c_n \leqq \alpha.$$

Hence we have

$$\measuredangle\ a_n b_n c_n < \pi - \measuredangle a_n c_n b_n \leqq \pi - \frac{\pi - \alpha}{2} = \frac{\pi + \alpha}{2}$$

and also $\measuredangle a_n c_n b_n < \frac{\pi + \alpha}{2}$. Then we can deduce

$$\max \measuredangle (a_n, b_n, c_n) < \frac{\pi + \alpha}{2},$$

which contradicts condition (i). This completes the proof of the sufficiency of Theorem 3.

Proof of the necessity. Let S be a compact set in $\mathbf{R}^2$, which is the boundary of a smooth strictly convex set. The necessity of condition (ii) follows from Theorem 2 in [6]. In order to prove the necessity of condition (i) we shall use the mapping $f: S \to C$ of the set S onto the unit circle defined by parallel supporting lines.

Let $p \in S$ and $\varepsilon > 0$ be a given positive number. We denote by d_p the unique supporting line of conv S through the point p and with d'_p the line tangent to the unit circle, which is parallel to d_p and relative to which C is on the same side as conv S relative to d_p. Let $f(p)$ be the unique contact-point of d'_p with the circle C. By a well-known theorem (see for instance [2] p. 13) $f: S \to C$ is a continuous mapping. Let q_1 and q_2 be the two points on the circle C such that the arcs $f(p)q_1$ and $f(p)q_2$ have the length ε. By the continuity of the mapping $f: S \to C$ there is a neighborhood U of p such that for every $x \in S \cap U$, $f(x)$ belongs to the open arc q_1q_2 (i.e. the arc q_1q_2 without its endpoints q_1 and q_2). There is then a $\delta(p, \varepsilon) > 0$ such that for every $x \in S \cap D(p, \delta(p, \varepsilon))$ $f(x)$ belongs to the open arc q_1q_2. Let r, s, t be three distinct points of $S \cap D(p, \delta(p, \varepsilon))$. We can suppose without loss of generality that the points r, s, t are in this order on the arc $S \cap D(p, \delta(p, \varepsilon))$. Let u be the intersection point of the supporting lines d_r and d_t of conv S through r and respectively through t. Let d'_r be the tangent line through $f(r)$ to the circle C (hence d_r is parallel to this tangent line) and d_t' be the tangent to the unit circle through $f(t)$ (hence parallel to d_t). Denote with v the intersection point of the lines d_r' and d_t'. Since s is in the interior of the triangle rut we have the inequality $\measuredangle rst > \measuredangle rut$ and therefore

$$\max \measuredangle (r,s,t) \geqq \measuredangle rst > \measuredangle rut = \measuredangle f(r)\, vf(t) > \pi - \varepsilon.$$

The last of these inequalities follows from the fact that $f(r)$ and $f(t)$ belong to the arc $q_1 f(p) q_2$ of length 2ε. With this we have proved the necessity of condition (i).

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