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ON AN ITERATIVE METHOD FOR SOLVING NONLINEAR EQUATIONS IN ORDERED BANACH SPACES

by
FLORIAN-ALEXANDRU POTRA
(Bucureşt)

JU. D. FEDORENKO [2] has proposed a simple iterative procedure of the form

(1) $$x_{n+1} = x_n - P(x_n)/M, \quad n = 0,\ 1,\ 2,\ \ldots$$

for solving equations in real Hilbert spaces. But his proof was false, the iterative procedure (1) being not convergent in the conditions stated by the author. One can easily construct examples of operators from $\mathbf{R}^2$ into $\mathbf{R}^2$ which satisfy the hypotheses of Fedorenko's theorem, and for which the procedure (1) is not convergent (see [4]). The purpose of this paper is to give sufficient conditions for the convergence of the iterative procedure (1), in the case where P is an operator which maps an ordered Banach space X into itself.

Let X be a Banach, space, and let us denote by θ its origin. We suppose that the space X is partially ordered by a relation „ $\prec$", which satisfies the following properties:

A1. $x \prec x$, for any $x \in X$.

A2. If $x \prec y$, and $y \prec z$, then $x \prec z$.

A3. If $x \prec y$, and $y \prec x$, then $x = y$.

A4. If $x \prec y$, then $x + z \prec y + z$, for any $z \in X$.

A5. If $x \prec y$, then $\lambda x \prec \lambda y$. for any nonegative number λ.

A6. If $(x_n)_{n=1}^{\infty}$ is a convergent sequence of elements of X, such that $\theta \prec x_n$, for $n = 1, 2, \ldots$, then $\theta \prec \lim\limits_{n\to\infty} x_n$.

A7. There exists a positive number q, such that $||x|| \leq q||y||$, for any pair (x, y) of elements of X having the property $\theta \prec x \prec y$.

It is easy to see that a binary relation satisfying the above properties can be induced by a normal cone (see [3] for the definition of the normal cone). We say that two elements x and y of X are comparable, if either $x \prec y$ or $y \prec x$ holds.

Now, let x_0 be a point belonging to the domain of P, such that $P(x_0)$ is comparable to θ, and let h be a positive number. We denote by H the set $\{x \in X;\ x \prec x_0,\ \|x - x_0\| \leq h\}$, if $\theta \prec P(x_0)$, or the set $\{x \in X;\ x_0 \prec x,\ \|x - x_0\| \leq h\}$, if $P(x_0) \prec \theta$.

THEOREM *If there exist two positive numbers* m *and* M, *suc hthat*

(2) $$m(x - y) \prec P(x) - P(y) \prec M(x - y)$$

for any x, y *of* H, *with* $x \prec y$, *and if*

(3) $$\|P(x_0)\| \leq \frac{mh}{2},$$

then by the iterative procedure (1), *one obtains a sequence* $(x_n)_{n=1}^{\infty}$, *of elements of* H, *converging to a root* x^* *of the equation* $P(x) = \theta$, *such that, for any* $n \in \{0, 1, 2, \ldots\}$, *the following inequalities are satisfied*:

(4) $$\|x - x^*\| \leq \frac{q}{m} \|P(x_0)\|\alpha^m,$$

(5) $$\|x_n - x^*\| \leq q \frac{M - m}{m} \|x_n - x_{n-1}\|,$$

where α *denotes the number* $\frac{M - m}{M}$.

Proof. We shall analize only the case $\theta \prec P(x_0)$, because the case $P(x_0) \prec \theta$ may be treated similarly. We shall prove, by induction, the following inequalities:

(6) $$x_n \prec x_0, \quad n = 0, 1, 2, \ldots$$

(7) $$\|x_n - x_0\| \leq h, \quad n = 0, 1, 2, \ldots$$

(8) $$\theta \prec x_n - x_{n+1} \prec \alpha^n(x_0 - x_1), \quad n = 0, 1, 2 \ldots$$

The above inequalities are obvious for $n = 0$. Let us suppose, that they are true for $n = 0, 1, \ldots, k$. From (6) and (8) it follows that

(9) $$x_{k+1} \prec x_k \prec x_0,$$

and thus (6) is true for $n = k + 1$.

Using (8), the above inequality, and the properties of the relation $\prec$, we have succesively:

$$\theta \prec x_0 - x_{k+1} = \sum_{i=0}^{k} (x_i - x_{i+1}) \prec (x_0 - x_1) \sum_{i=1}^{k} \alpha^i \prec \frac{x_0 - x_1}{1 - \alpha} = \frac{P(x_0)}{m},$$

wherefrom we infer that (9) is true for $n = k + 1$.

Now, as x_k, $x_{k+1} \in H$, and $x_{k+1} \prec x_k$, we have

$$m(x_k - x_{k+1}) \prec P(x_k) - P(x_{k+1}) \prec M(x_k - x_{k+1}),$$

and substracting $m(x_k - x_{k+1})$ from each term of the above relation we obtain

(10) $$\theta \prec P(x_{k+1}) \prec (M - m)(x_k - x_{k+1}).$$

This relation, together with (1), imply that:

(11) $$\theta \prec x_{k+1} - x_{k+2} = \frac{P(x_{k+1})}{M} \prec \alpha(x_k - x_{k+1}).$$

From the above inequality if follows that (8) is true for $n = k + 1$. Thus, according to the induction principle, the inequalities (6)—(8) are true for any nonnegative integer n. The inequalities (6) and (7) imply that $x_n \in H$ for $n = 0, 1, 2, \ldots$ Using (8) we can write

$$x_n - x_{n+p} = \sum_{i=0}^{p-1} (x_{n+i} - x_{n+i+1}) \prec \frac{(x_0 - x_1)\alpha^n}{1 - \alpha} = \frac{P(x_0)}{m} \cdot \alpha^n,$$

wherefrom we obtain the inequality

(12) $$\|x_n - x_{n+p}\| \leq q \frac{\|P(x_0)\|}{m} \alpha^n,$$

which shows that the sequence $(x_n)_{n=1}^{\infty}$ is a fundamental one. The space X being complete, there exists an element x^* of X, such that $x^* = \lim_{n\to\infty} x_n$. From (6) and A6, it follows that $x^* \prec x_0$, while from (7) it results that $\|x_0 - x^*\| \leq h$. Thus $x^* \in H$.

The inequality (10) implies that $x_{n+p} \prec x_n$ for any positive integers n and p. so that for $p \to \infty$ we obtain:

(13) $$x^* \prec x_n, \quad n = 0, 1, 2, \ldots$$

Now, by (2), we have

$$\theta \prec m(x_n - x^*) \prec P(x_n) - P(x^*) \prec M(x_n - x^*)$$

and hence, by virtue of A7,

$$\|P(x_n) - P(x^*)\| \leq qM\|x_n - x^*\|.$$

The above inequality implies that $P(x^*) = \lim_{n\to\infty} P(x_n)$, while from (10) it follows that $\lim_{n\to\infty} P(x_n) = \theta$. Thus x^* is a root of the equation $P(x) = \theta$. The inequality (4) can be obtained by letting $p \to \infty$ in (12). On the other hand, from (11), we have

$$\theta \prec x_n - x_{n+p} \prec \frac{\alpha}{1-\alpha}(x_{n-1} - x_n) = \frac{M-m}{m}(x_{n-1} - x_n).$$

According to A7, the above relation implies the inequalitiy (5), and so the proof of our theorem is completed.

Notes. a) If the relation „$\prec$" is not supposed to satisfy condition A6, but in exchange we suppose that the operator P is continous in the sphere $B(x_0, h) = \{x \in X ; ||x - x_0|| \leq h\}$, the above theorem remains true, with the exception of the fact that in the conclusion of the theorem we must replace the relation $x^* \in H$ by the relation $x^* \in B(x_0, h)$.

b) From (2), it follows that in H there exists no other root of the equation $P(x) = \theta$, comparable to x^*, so that, in the particular case where $\prec$ is a total order relation, x^* is the unique root of the equation $P(x) = \theta$, in the set H.

c) In the same case, when $\prec$ is a total order relation, the condition (3) implies the continuity of the operator P in the set H.

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Concluding, let us make some considerations on the method of approximate solving of equations, presented above. It can be observed that the disadvantage of a low order of convergence is compensated by the particular simplicity of the algorithm, and by the fact that the condition (3) imposed to the initial aproximation, in order to assure the convergence of the iterative procedure, is very weak. This condition is as weaker as h is larger.

In the particular case when $h \to \infty$, the only condition required from x_0 is that its image trough P be comparable to θ in the sense of the order relation.

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INCREST
Bucureşti