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ON A MODIFIED SECANT METHOD

by

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Abstract. In this paper we apply the metod of V. PTÁK ([4], [5]) to the study of the convergence of a modified secant method. We prove that the rate of convergence of this method is of the form

$$\omega(r) = \frac{r}{d}\left(Hr + d - 2\sqrt{H^2a^2 - Hdr}\right)$$

where a, d, H and r are positive numbers depending on the initial conditions. We also give sharp estimates for the distance $\|x_n - x^*\|$, $n = 1, 2, \ldots$, where $(x_n)_{n=1}^{\infty}$ is the sequence obtained by the modified secant method and x^* is its limit.

1. The Induction Theorem

The method of Nondiscrete Mathematical Induction, introduced by V. PTÁK [4], has allowed a new approach in the study of the convergence of iterative procedures. An important role in this approach is played by the notion of the rate of convergence [5], [6]. Let T be an interval of the form $T = \{r \in \mathbf{R}\,;\ 0 < r \leq r_0\}$, for some positive r_0 (i.e. $T =]0, r_0]$). Let ω be a function defined on T. We define by reccurence:

$$\omega^0(r) = r,\ \omega^{n+1}(r) = \omega(\omega^n(r)), \qquad n = 0, 1, 2, \ldots$$

DEFINITION 1.1. *The function ω, defined on T, is called a rate of convergence, if it satisfies the following properties:*

(1) *ω maps T into itself;*

(2) *for each $r \in T$ the series $\sum_{n=0}^{\infty} \omega^n(r)$ is convergent.*

The sum of the above series, $\sigma(r) = \sum_{n=0}^{\infty} \omega^n(r)$, obviously satisfies the following functional equation:

(3) $$\sigma(r) = r + \sigma(\omega(r)).$$

We shall justify the name of „rate of convergence", given to the function ω, after stating the Induction Theorem.

Let (X, d) be a complete metric space. If A is a subset of X, and x an element of X, we shall denote by $d(x, A)$ the g.l.b. of the set $\{d(x, y); y \in A\}$. For any positive number r we shall denote by $U(A, r)$ the set $\{x \in X; d(x, A) \leq r\}$. If x is an element of X, we shall write for simplicity $U(x, r)$ instead of $U(\{x\}, r)$.

Let us denote by T the interval $]0, r_0]$ of the real line, and for each $r \in T$, let $Z(r)$ represent a certain subset of X. We shall use the following notation for the limit of the family $Z(.)$.

(4) $$Z(0) = \bigcap_{s>0} \bigcup_{r<s} Z(r)^-.$$

Now, we can state the Induction Theorem [4].

THEOREM 1.1. *If*

(5) $$Z(r) \subset U(Z(\omega(r)), r).$$

for each $r \in T$, *then*

(6) $$Z(r) \subset U(Z(0), \sigma(r)),$$

for each $r \in T$.

We shall sketch below how the method of nondiscrete mathematical induction can be applied to the study of the convergence of iterative procedures. Let F be a mapping of the complete metric space X into itself, and let x_0 be an element of X. Suppose that we can attach to the pair (F, x_0) a rate of convergence ω on the interval $T =]0, r_0]$, and a family of sets $\{Z(r)\}_{r \in T}$, such that the following relations be fulfiled:

(7) $$x_0 \in Z(r_0),$$

(8) $$x \in Z(r) \Rightarrow F(x) \in U(x, r) \cap Z(\omega(r)) \text{ for each } r \in T.$$

Then the Induction Theorem assures the fact that $Z(0) \neq \emptyset$. On the other hand (8) implies that each element ξ of $Z(0)$ is a fixed element of the mapping F i.e. $F(\xi) = \xi$. It also follows that via the iterative procedure:

(9) $$x_{n+1} = F(x_n), \; n = 0, 1, 2, \ldots,$$

we obtain a sequence $(x_n)_{n=0}^{\infty}$ which converges to an element $x^* \in Z(0)$, such that the following inequalities are satisfied:

(10) $$d(x_{n+1}, x_n) \leq \omega^n(r_0), \; n = 0, 1, 2, \ldots$$

(11) $$d(x_n, x^*) \leq \sigma(\omega^n(r_0)) \; n = 0, 1, 2, \ldots$$

From (10) one obtains the following estimates of the distance between the n' th iterate x_n and the „starting point" x_0:

(12) $$d(x_n, x_0) \leq \sigma(r_0) - \sigma(\omega^n(r_0)).$$

The relation (11) will be called an apriori estimate fo the distance between the n' th iterate given by the procedure (9) and the fixed point x^*. The name „apriori estimate" is justified by the fact that one can compute this estimate before performing the iterative procedure.

Suppose, that for a certain $n \in \{1, 2, \ldots\}$, one has already computed $x_1, x_2, \ldots, x_n$. If

(13) $$x_{n-1} \in Z(d(x_n, x_{n-1})),$$

then it can easily be proved that the following inequality is satisfied:

(14) $$d(x_n, x^*) \leq \sigma(\omega(d(x_n, x_{n-1})) = \sigma(d(x_n, x_{n-1})) - d(x_n, x_{n-1}).$$

The above estimate will be called an „aposteriori estimate", because it can be computed only after performing the iterative procedure (9). The aposteriori estimates are generally better than the apriori ones.

Summing up what we have stated above, we get the following:

Corollary. If the conditions (7) and (8) are satisfied, then by the iterative procedure (9) one obtains a sequence $(x_n)_{n=0}^{\infty}$ which converges to a fixed point x^* of the mapping F, and for each $n \in \{0, 1, 2, \ldots\}$ the inequalities (10)–(12) are fulfiled. Moreover, if for a certain $n \in \{1, 2, 3, \ldots\}$ the condition (13) is satisfied, then for this n, the inequality (14) is also fulfiled.

The above *corollary* will be the basis of the prof of the Theorem 3.1, concerning the convergence of the modified secant method, which will be given in Section 3.

2. Divided differences of an operator

The notion of divided difference of a (nonlinear) operator is an extension of the usual notion of divided difference of a function, in the same sense in which the Fréchet derivative of an operator is an extension of the classical notion of the derivative of a function. This notion was introduced by J. SCHRODER [8] and was used by A. SERGEEV [9] and J. SCHMIDT [7] to the extension of the secant method for the iterative solution of the nonlinear operatorial equations in Banach spaces.

Let E and F be two Banach spaces. We shall denote by $L(E, F)$ the Banach space of all linear and bounded operators, from E into F. Let f be a (nonlinear) operator from E into F, and let x and y be two different points of the domain of f.

DEFINITION 2.1. A bounded linear operator $A \in L(E, F)$ is called a divided difference of the operator f on the points x and y, if:

(15) $$A(x - y) = f(x) - f(y).$$

In the scalar case the divided difference of a function is unique, but in the general case this assertion is not true. Let us examine as an illustration the case where $E = F = R^2$. In this case, a nonlinear operator f is characterized by two real functions of two real variables f_1 and f_2 i.e.

$$(\forall)\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^2, \quad f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}.$$

Then each of the linear operators A_1 and A_2 given by the following two matrices satisfy (15):

$$A_1 = \begin{pmatrix} \dfrac{f_1(x_1, y_2) - f_1(y_1, y_2)}{x_1 - y_1} & \dfrac{f_1(x_1, x_2) - f_1(x_1, y_2)}{x_2 - y_2} \\ \dfrac{f_2(x_1, y_2) - f_2(y_1, y_2)}{x_1 - y_1} & \dfrac{f_2(x_1, x_2) - f_2(x_1, y_2)}{x_2 - y_2} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \dfrac{f_1(x_1, x_2) - f_1(y_1, x_2)}{x_1 - y_1} & \dfrac{f_1(y_1, x_2) - f_1(y_1, y_2)}{x_2 - y_2} \\ \dfrac{f_2(x_1, x_2) - f_2(y_1, x_2)}{x_1 - y_1} & \dfrac{f_2(y_1, x_2) - f_2(y_1, y_2)}{x_2 - y_2} \end{pmatrix}$$

If f is differentiable and its Fréchet derivatives f' is continuous on the segment $[x, y] = \{tx + (1-t)y\,;\ t \in [0, 1]\}$, then the linear operator given by

$$A_3 = \int_0^1 f'(x + t(y - x))dt,$$

also satisfies (15). That means that any of the three linear operators A_1, A_2, A_3, are devided differences of the operator f on the points x and y. Moreover, any convex combination of A_1, A_2 and A_3 is also a divided difference of f on the points x and y. It we have two divided differences of f on the points x and y, represented by the matrices A and B, then the matrix, C, having the first line equal to the first line of A, and the second line equal to the second line of B, also represents a divided difference of f on the points x and y.

Let us now return to the general case. Concerning the existence of the divided differences see [1]. Concerning other examples in some concrete spaces see [10]. Let us suppose that the closed sphere $U = U(x_0, m)$ is included into the domain of the operator f, and let us denote by D the set $D = \{(x, y) \in U \times U\,;\ x \neq y\}$. We consider the mapping:

$$D \ni (x, y) \to [x, y\,;\ f] \in L(E, F)$$

where, for any pair $(x, y) \in D$, the linear operator $[x, y\,;\ f]$ is a divided difference of f on the points x and y i.e.:

$$(16) \qquad [x, y\,;\ f](x - y) = f(x) - f(y)$$

In [9] one assumes that the mapping $(x, y) \to [x, y\,;\ f]$ is symmetric i.e. $[x, y\,;\ f] = [y, x\,;\ f]$. In [7] this condition is no longer required. Let us remark that in our example A_1 and A_2 are not symmetric, while A_3 and $\frac{1}{2} A_1 + \frac{1}{2} A_2$ are.

In both of the above cited papers, one supposes, in order to assure sufficient conditions for the convergence of the secant method, that the mapping $(x, y) \to [x, y\,;\ f]$ satisfies a Lipschitz condition at least. We shall write this condition under the form:

$$(17) \qquad \|[x, y\,;\ f] - [u, v\,;\ f]\| \leqq H(\|x - u\| + \|y - v\|).$$

It is easy to prove that if the above inequality is fulfiled for all $x, y, u, v \in U = U(x_0, m)$, with $x \neq y$ and $u \neq v$, then for each $x \in U$ there exists the limit $\lim_{y \to x} [x, y\,;\ f]$, and it equals the Fréchet derivative $f'(x)$. We have then:

$$(18) \qquad \|f'(x) - f'(y)\| \leqq 2H\|x - y\|, \quad x, y \in U$$

The above remark allows us to take by definition $[x, x\,;\ f] = f'(x)$ for each $x \in X$. Thus (18) implies (17).

Reversely, if the operator f is Fréchet differentiable for each $x \in U$, and if (18) is satisfied, then there exists a mapping $U \times U \ni (x, y) \to [x, y\,;\ f] \in L(E, F)$ which satisfies(16) and (17). We can take, for example,

$$[x, y\,;\ f] = \int_0^1 f'(x + t(y - x))dt.$$

This remark will be used to obtain the theorem concerning the convergence of the modified Newton's process [3] as a consequence of the theorem concerning the convergence of the modified secant method wich will be proved in the next section.

3. The modified secant method

The same as in the preceding section, let f be a nonlinear operator from the Banach space E into the Banach space F, and let the sphere $U = U(x_0, m)$ be included into its domain of definition. We suppose that there exists a mapping.

$$U \times U \ni (x, y) \to [x, y\,;\ f] \in L(E, F).$$

which satisfies (16) and (17). Let $\bar{x}_0$ be a point of U, for which the linear operator $[x_0, \bar{x}_0\,;\ f]$ is boundedly invertible. The modified secant method, we are going to study, consists of the following interative procedure:

$$(19) \qquad x_{n+1} = x_n - [x_0, \bar{x}_0\,;\ f]^{-1} f(x_n), \quad n = 0, 1, 2, \ldots$$

For the study of the convergence of the sequence $(x_n)_{n=0}^{\infty}$ yielded by (19), we need some results concerning the behaviour of such a sequence in the particular case where f is a certain real quadratic polinomial.

LEMMA 3.1. *If d, H, q_0 and r_0 are positive numbers satisfying the conditions*

$$(\sqrt{r_0}+\sqrt{q_0+r_0})^2 \leq \frac{d}{H} \tag{20}$$

then the function

$$\omega(r)=\frac{r}{d}\left(Hr+d-2\sqrt{H^2a^2+Hdr}\right) \tag{21}$$

is a rate of convergence on the interval $T=]0,\ r_0]$, and the corresponding function σ is given by

$$\sigma(r)=\sqrt{a^2+\frac{d}{H}r}-a, \tag{22}$$

where,

$$a=\frac{1}{2H}\sqrt{(d-Hq_0)^2-4Hdr_0} \tag{23}$$

Proof. First, we observe that the inequality (20) implies that the quantity under the square root sign from (23) is nonnegative. Let us consider the real polinominal

$$f(x)=H(x^2-a^2). \tag{24}$$

It is easy to prove, that for any starting point x_0, chosen in the interval $]a, +\infty[$, and for any positive number d, belonging to the interval $[f'(x_0), +\infty[$, the iterative procedure

$$x_{n+1}=x_n-f_n(x)/d \tag{25}$$

yields a sequence $(x_n)_{n=0}^{\infty}$, decreasingly converging to the root $x^*=a$ of the equation $f(x)=0$.

Setting for any $r\in\ ]0,\ r_0]$

$$x_0=x_0(r)=\sqrt{a^2+\frac{d}{H}r}\,, \tag{26}$$

we have $x_0>x^*$, and $f(x_0)/d=r$. Taking $\omega(r)=f(x_1)/d$ and $\sigma(r)=x_0-x^*$ we obtain the formulas (22) and (23).

Denoting $\bar{x}_0=x_0(r_0)+q_0$, and computing the divided difference of the function f on the points $x_0(r_0)$ and $\bar{x}_0$ we obtain

$$[x_0,\ \bar{x}_0;\ f]=d. \tag{27}$$

Taking into account the fact that f is a convex function, we infer that

$$d\geq f'(x_0(r_0))\geq f'(x_0(r)) \text{ for any } r\in\ ]0,\ r_0].$$

Thus, for each $r\in\ ]0,\ r_0]$, we shall obtain, via the iterative procedure (25), a sequence $(x_n)_{n=0}^{\infty}$, decreasingly converging to x^*. In this case it is clear that the functions ω and σ, defined as above, represent a rate of convergence and the function related to it. The following equalities are obviously satisfied:

$$x_0-x_n=\sigma(r)-\sigma(\omega^n(r)), \tag{28}$$

$$x_n-x_{n+1}=\omega^n(r), \tag{29}$$

$$x_n-x^*=\sigma(\omega^n(r)). \tag{30}$$

Now, we are able to state our result concerning the modified secant method:

THEOREM 3.1. *If the conditions* (16) *and* (17) *are satisfied for all x, y, u, $v\in U=U(x_0,\ m)$, and if the following inequalities:*

$$\|[x_0,\ \bar{x}_0;\ f]^{-1}\|^{-1}\geq d, \tag{31}$$

$$\|x_0-\bar{x}_0\|\leq q_0, \tag{32}$$

$$\|[x_0,\ \bar{x}_0;\ f]^{-1}f(x_0)\|\leq r_0, \tag{33}$$

$$(\sqrt{r_0}+\sqrt{q_0+r_0})^2\leq\frac{H}{d}, \tag{34}$$

$$m\geq\sigma(r_0), \tag{35}$$

are fulfiled, then the sequence $(x_n)_{n=0}^{\infty}$, obtained by the iterative procedure (19), *converges to a root x^* of the equation $f(x)=0$, and the following inequalities are satisfied:*

$$\|x_n-x_0\|\leq\sigma(r_0)-\sigma(\omega^n(r_0)),\quad n=0,1,2,\ldots, \tag{36}$$

$$\|x_n-x^*\|\leqslant\sigma(\omega^n(r_0)),\quad n=0,1,2,\ldots, \tag{37}$$

$$\|x_n-x^*\|\leq\sigma(\|x_n-x_{n-1}\|)-\|x_n-x_{n-1}\|,\quad n=1,2,3,\ldots, \tag{38}$$

where ω and σ are given respectively by (22) *and* (23).

Proof. The proof is based on the Corollary stated in Section I and on the Lemma 3.1 proved in the present section. The iterative procedure (19) is of the form (7) with $F(x)=x-[x_0,\ \bar{x}_0;\ f]^{-1}f(x)$, for $x\in U$. Taking into account the inversability of $[x_0,\ \bar{x}_0;\ f]$, it follows that every fixed point of F is a root of the equation $f(x)=0$. We attach to the pair (F, x_0) the rate of convergence ω given by (22) and the family of sets:

$$Z(r) = \{x \in E;\ \|[x_0, \bar{x}_0; f]^{-1} f(x)\| \leq r,\quad \|x - x_0\| \leqslant \sigma(r_0) - \sigma(r)\},\quad r \in]0, r_0] \tag{39}$$

It is clear that $z(r_0) = \{x_0\}$, so that condition (7) of the above mentioned Corollary is satisfied. We shall prove that condition (8) is also satisfied. Let x be an element of $z(r)$, and let

$$x' = F(x) = x - [x_0, \bar{x}_0; f]^{-1} f(x). \tag{40}$$

Using (3) we can write

$$\|x' - x_0\| \leq \|x' - x\| + \|x - x_0\| \leqslant r + \sigma(r_0) - \sigma(r) = \sigma(r_0) - \sigma(\omega(r)). \tag{41}$$

From (16) and (40) we infer that

$$f(x') = f(x') - f(x) - [x_0, \bar{x}_0; f](x' - x) = ([x', x; f] - [x_0, \bar{x}_0; f])(x' - x).$$

According to the conditions (17), (31) and (32), the above equality yields:

$$\|[x_0, \bar{x}_0; f]^{-1} f(x')\| \leq \frac{H}{d} (2\|x - x_0\| + \|x' - x\| + \|x_0 - \bar{x}_0\|)\|x' - x\|.$$

Using (22), (23), (39) and (40), we obtain

$$\|[x_0, \bar{x}_0; f]^{-1} f(x')\| \leqslant \omega(r).$$

This relation together with (41) imply that $x' \in Z(\omega(r))$ so that condition (6) is also fulfiled. It follows that by the iterative procedure (19), one obtains a sequence $(x_n)_{n=0}^{\infty}$ which converges to a root x^* of the equation $f(x) = 0$. Moreover for each $n \in \{0, 1, 2, \ldots\}$ the inequalities (10)—(12) are satisfied. But the inequalities (11) and (12), correspond respectively to the inequalities (37) and (36), while from (10), (12), and from the fact that σ is an increasing function on $]0, r_0]$ we infer that

$$\|x_{n-1} - x_0\| \leq \sigma(r_0) - \sigma(\|x_n - x_{n-1}\|),\qquad n = 1, 2, 3, \ldots$$

The above relation shows that $x_{n-1} \in Z(\|x_n - x_{n-1}\|)$ for $n = 1, 2, 3, \ldots$ so that the condition (13) of the Corollary is fulfiled. Consequently the aposteriori estimate (38), which correspond to the inequality (14), will be satisfied for $n = 1, 2, 3, \ldots$.

Concerning the hypotheses of the above theorem, we have to note that, in practical applications, the number q_0 from the left side of the inequality (32) can be taken as small as wanted, because having an initial approximation x_0, one can take for $\bar{x}_0$ a small perturbation of it (for example $\bar{x}_0 = (1 + \varepsilon)x_0$). The key condition of our theorem is represented by the inequality (34). This inequality can be satisfied only if r_0 is small enogüh, that is, if the initial approximation x_0 is good enough. However, we can prove that the condition (34) is in some sense the weakest possible. Indeed, let d, H, q_0 and r_0 be some positive numbers, and let us consider the real function f given by the formula

$$f(x) = Hx^2 - dr_0 - \frac{1}{4H}(d - Hq_0)^2.$$

The divided difference of the function f, will obviously satisfy (16) and (17). The inequalities (31)—(33) are also verified, if we take

$$x_0 = \frac{d - Hq_0}{2H},\qquad \bar{x}_0 = \frac{d + Hq_0}{2H}.$$

However, if the condition (34) is not verified, then $dr_0 > \frac{1}{4H}(d - Hq_0)^2$, and thus the equation $f(x) = 0$ has no solution.

In the following we shall show that the estimates (36)—(38), obtained in Theorem 3.1, are, in some sense, the best possible.

PROPOSITION 3.1. *The estimates* (36)—(38) *are sharp in the following sense: for any positive numbers d, H, q_0 and r_0, satisfying the inequality* (34), *there exists a function f and a pair of points $(x_0, \bar{x}_0)$ which satisfy the hypothesis of Theorem* 3.1, *and for which the inequalities* (36)—(38) *are verified with equality.*

Proof. The proof of the above proposition is a consequence of the proof of Lemma 3.1.

From (36) it follows that $\|x^* - x_0\| \leq \sigma(r_0)$. We shall prove that x^* is the unique root of the equation $f(x) = 0$ in a neighbourhood of the point x_0. Let V denote the open sphere with centre x_0 and radius $\sigma(r_0) + 2a$.

PROPOSITION 3.2. *If the inequality* (34) *from Theorem* 3.1 *is strict, then the root x^*, whose existence is guaranteed by this theorem, is the unique solution of the equation $f(x) = 0$ in the set $U \cap \mathring{V}$*

Proof. First, we note that if the inequality (34) is strict, the $a > 0$, so that $x^* \in U \cap \mathring{V}$. Let Y^* be an element of $U \cap \mathring{V}$, such that $f(y^*) = 0$. Using (16) we obtain the relation:

$$x^* - y^* = [x_0, \bar{x}_0; f]^{-1}([x_0, \bar{x}_0; f] - [x^*, y^*; f])(x^* - y^*). \tag{41}$$

Now taking into account (17) we obtain:

$$\|x^* - y^*\| \leqslant \frac{H}{d}(\|x_0 - x^*\| + \|\bar{x}_0 - y^*\|)\|x^* - y^*\|. \tag{42}$$

On the other hand, from (22), (31) and (32), we infer that

$$(43)\quad \frac{H}{d}(\|x_0 - x^*\| + \|\bar{x}_0 - y^*\|) < \frac{H}{d}(2\sigma(r_0) + 2a + q_0) = 1$$

Finally the inequalities (42) and (43) imply that $x^* = y^*$, so that the proof of the proposition is completed.

4. The modified Newton's method

As we have anticipated in Section 2, the results concerning the modified Newton's method can be obtained, as a limit case, from the results concerning the modified secant method. In the following, we shall transcribe the results obtained in the preceding section for the case where $x_0 = \bar{x}_0$ and $q_0 = 0$.

LEMMA 4.1. *If d, H and r_0 are three positive numbers satisfying the inequality;*

$$(44)\quad 4Hr_0 \leq d,$$

then:

$$(45)\quad \omega_1(r) = \frac{r}{d}\left(Hr + d - \sqrt{d^2 - 4Hd(r_0 - r)}\right)$$

is a rate of convergence on the interval $T =]0, r_0]$ and the corresponding function σ_1 is given by:

$$(46)\quad \sigma_1(r) = \frac{1}{2H}\left(\sqrt{d^2 - 4Hd(r_0 - r)} - \sqrt{d^2 - 4Hdr_0}\right).$$

Now, as in the preceeding two sections, let f be a nonlinear operator which maps the sphere $U = U(x_0, m)$ of the Banach space E into the Banach space F. We suppose that f is Fréchet differentiable on U and that the condition (18) holds. Then, according to the remark made in Section 2, there exists a mapping

$$U \times U \ni (x, y) \mapsto [x, y; f] \in L(E, F)$$

such that (16) and (17) hold. Moreover for each $x \in U$ we have $[x, x; f] = f'(x)$.

Let us suppose now that the Fréchet derivative $f'(x_0)$ is boundedly invertible. We may then consider the following iterative procedure:

$$(47)\quad x_{n+1} = x_n - [f'(x_0)]^{-1}f(x_n), \quad n = 0, 1, 2, \ldots$$

which is called the modified Newton's method. This procedure may be regarded as a limit case of the modified secant method so that from Theorem 3.1 we can derive the following theorem:

THEOREM 4.1. *If condition (18) holds for each $x, y \in U = U(x_0, m)$ and if the following inequalities:*

$$(48)\quad \|[f'(x_0)]^{-1}\|^{-1} \geq d$$

$$(49)\quad \|[f'(x_0)]^{-1}f(x_0)\| \leq r_0,$$

$$(50)\quad 4Hr_0 \leq d,$$

$$(51)\quad m \geq \sigma_1(r_0) = \frac{1}{2H}\left(d - \sqrt{d^2 - 4Hdr_0}\right),$$

are fulfiled, then the sequence $(x_n)_{n=0}^{\infty}$ obtained by the iterative procedure (47), converges to a root x^* of the equation $f(x) = 0$, and the following inequalities are satisfied:

$$(52)\quad \|x_n - x_0\| \leq \sigma_1(r_0) - \sigma_1(\omega_1^n(r_0)), \qquad n = 0, 1, 2, \ldots,$$

$$(53)\quad \|x_n - x^*\| \leq \sigma_1(\omega_1^n(r_0)), \qquad n = 0, 1, 2, \ldots,$$

$$(54)\quad \|x_n - x^*\| \leq \sigma_1\|(x_n - x_{n-1}\|) - \|x_n - x_{n-1}\|, \qquad n = 1, 2, 3, \ldots,$$

where ω_1 and σ_1 are given respectively by (45) and (46).

From Propositions 3.1 and 3.2 we obtain the following two propositions, concerning the sharpness of the estimates (58)—(60) and the uniqueness of the root x^*:

PROPOSITION 4.1. *The estimates (52)—(54) are sharp in the following sense: For any three positive numbers d, H and r_0 satisfying the inequality (50) there exists a function f, which satisfies the hypotheses of Theorem 4.1, and for which the inequalities (52)—(54) are verified with equality.*

PROPOSITION 4.2. *If the inequality (50) of Theorem 4.1 is strict, then the root x^*, whose existence is guaranteed by Theorem 4.1, is the unique solution of the equation $f(x) = 0$ in the set $U \cap \overset{\circ}{V}$, where $\overset{\circ}{V}$ is the open sphere with center x_0 and radius $\sigma_1(r_0) + 2a$.*

In the end, let us note that the results stated in this section represent a slight improvement of the results obtained by us in [3]. Nanemely the condition (18) of the present paper in weaker than the condition $\|f''(x)\| \leq 2H$, $x \in U$, imposed there. Moreover the aposteriori estimate (54), from Theorem 4.1, is new.

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