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ON INTERPOLATION OPERATORS (II) (A PROOF OF TIMAN'S THEOREM FOR DIFFERENTIABLE FUNCTIONS)

by

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1. In continuation of our study of interpolation operators which respectively satisfy the inequalities of JACKSON [2] TIMAN [6] and TELYAKOVSKII – GOPENGAUZ [5, 1] for differentiable functions, in this paper, we propose to construct interpolation operators $Q_{nr}(f, x)$ which satisfy Timan's inequality for functions $f(x) \in C^1[-1, 1]$. Our earlier work [4] was devoted to constructing and studying the interpolation operators which satisfy Jackson's inequality for functions $f(x) \in C^1[-1, 1]$.

Let

$$-1 \leqslant x \leqslant 1, \ \cos t = x, \text{ and } \cos t_{kn} = x_{kn}$$

with

$$t_{kn} = \frac{2k\pi}{2n+1}, \quad k = \overline{0, n}\ ^*, \quad n = 1, 2, \ldots. \tag{1.1}$$

Further for $k = \overline{-n, n}$, let

$$l_{kn}(t) = \frac{\sin \frac{2n+1}{2}(t - t_{kn})}{(2n+1)\sin \frac{1}{2}(t - t_{kn})} \tag{1.2}$$

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* $k = \overline{0, n}$ stands for $k = 0, 1, 2, \ldots, n$.

and

$$(1.3)\qquad p_{kn}(t) = \frac{1}{43}\left[1008 l_{kn}^5(t) - 1820 l_{kn}^6(t) + 960 l_{kn}^7(t) - 105 l_{kn}^8(t)\right].$$

Then for any function $f(x)$ given on $[-1, 1]$ we define the operators

$$(1.4)\qquad Q_{nr}(f, x) = \left[\frac{1+x}{2} f(1) + \frac{1-x}{2} f(-1)\right] +$$

$$+ \sum_{k=0}^{n}\left[\sum_{\nu=0}^{r} (x - x_{kn})^{\nu} f^{(\nu)}(x_{kn}) - \left\{\frac{1+x}{2} f(1) + \frac{1-x}{2} f(-1)\right\}\right] \cdot q_{kn}(x)$$

for $r = 0$ and 1, where

$$(1.5)\qquad \begin{cases} q_{0n}(x) = p_{0n}(t) \\ q_{kn}(x) = p_{kn}(t) + p_{-kn}(t), \quad k = \overline{1, n}. \end{cases}$$

As in [4], we observe that $Q_{nr}(f, x)$ is an algebraic polynomial of degree $\leq 8n + 1$ in x interpolating the function $f(x)$ and its derivative at the points x_{kn}, $k = \overline{0, n}$.

We shall prove the following

THEOREM. *Let $f^{(r)}(x) \in C[-1, 1]$, then for the operators $Q_{nr}(f, x)$, we have*

$$(1.6)\qquad |Q_{nr}(f, x) - f(x)| \leq C_r \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^r \omega_{f^{(r)}}\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right),$$

$r = 0, 1$; $-1 \leq x \leq 1$. Here $\omega_{f^{(r)}}(\cdot)$ is the modulus of continuity of $f^{(r)}$ and C_r is an arbitrary positive constant.

The convergence properties as shown in (1.6) are due to structural properties of $Q_{nr}(f, x)$. The basis for the construction of $Q_{nr}(f, x)$ is to observe and obtain the identity

$$(1.7)\qquad \frac{1}{43}\sum_{k=-n}^{n}\left[1008 l_{kn}^5(t) - 1820 l_{kn}^6(t) + 960 l_{kn}^7(t) - 105 l_{kn}^8(t)\right] =$$

$$= 1 + \frac{14(n+2)(n+1)n(n-1)}{43(2n+1)^4}\left[10 - 15\cos(2n+1)t +\right.$$

$$\left. + 6\cos(4n+2)t - \cos(6n+3)t\right],$$

which, as we shall see, plays a vital role in these investigations. We will say that once the identity (1.7) is established, the work is half done. So in the following section we first prove (1.7).

2. Proof of the identity (1.7). Following KIS-VERTESI [3] (also c.f. [4]), we have for a positive integer m *

$$(2.1)\qquad (2n+1)^{m-1}\sum_{k=-n}^{n} l_{kn}^m(t) = C_{0,m} + 2\sum_{j=1}^{\left[\frac{mn}{2n+1}\right]} C_{(2n+1)j,m}\cos(2n+1)jt,$$

where the numbers $C_{j,m}$ satisfy

$$(2.2)\qquad \sum_{j=-mn}^{mn} C_{j,m} Z^j = Z^{-mn}(1 - Z^{2n+1})^m \sum_{j=0}^{\infty}\frac{(j+1)(j+2)\ldots(j+m-1)}{(m-1)} Z^j.$$

From (2.1), we see that

(2.3)

$$(2n+1)^4\sum_{k=-n}^{n} l_{kn}^5(t) = C_{0,5} + 2[C_{2n+1,5}\cos(2n+1)t + C_{4n+2,5}\cos(4n+2)t],$$

$$(2.4)\qquad (2n+1)^5\sum_{k=-n}^{n} l_{kn}^6(t) = C_{0,6} + 2[C_{2n+1,6}\cos(2n+1)t +$$

$$+ C_{4n+2,6}\cos(4n+2)t],$$

$$(2.5)\qquad (2n+1)^6\sum_{k=-n}^{n} l_{kn}^7(t) = C_{0,7} + 2[C_{2n+1,7}\cos(2n+1)t +$$

$$+ C_{4n+2,7}\cos(4n+2)t + C_{6n+3,7}\cos(6n+3)t],$$

$$(2.6)\qquad (2n+1)^7\sum_{k=-n}^{n} l_{kn}^8(t) = C_{0,8} + 2[C_{2n+1,8}\cos(2n+1)t +$$

$$+ C_{4n+2,8}\cos(4n+2)t + C_{6n+3,8}\cos(6n+3)t].$$

Thus to calculate the sums in (2.3)—(2.6), we need to calculate the numbers $C_{j,m}$ from (2.2) for j in multiples of $(2n+1)$. We have already shown in [4] that**

$$(2.7)\qquad C_{4n+3,5} = \frac{1}{4!}[(9n+3)_4 - 5(7n+2)_4 + 10(5n+1)_4 -$$

$$- 10(3n)_4 + 5(n-1)_4] = \frac{1}{4!}(n^4 + 2n^3 - n^2 - 2n),$$

* We describe the working in short only for completeness.

** $(j+m)_n = (j+m)(j+m+1)\ldots(j+m+n-1)$

$$(2.8)\quad C_{2n+1,5} = \frac{1}{4!}\,[(7n+2)_4 - 5(5n+1)_4 + 10(3n)_4 - 10(n-1)_4] = \frac{1}{4!}\,(76n^4 + 152n^3 + 104n^2 + 28n),$$

$$(2.9)\quad C_{0,5} = \frac{1}{4!}\,[(5n+1)_4 - 5(3n)_4 + 10(n-1)_4] = \frac{1}{4!}\,(230n^4 + 460n^3 + 370n^2 + 140n + 24),$$

$$(2.10)\quad C_{4n+2,6} = \frac{1}{5!}\,[(10n+3)_5 - 6(8n+2)_5 + 15(6n+1)_5 - 20(4n)_5 + 15(2n-1)_5] = \frac{2n+1}{5!}\,(16n^4 + 32n^3 + 14n^2 - 12n),$$

$$(2.11)\quad C_{2n+1,6} = \frac{1}{5!}\,[(8n+2)_5 - 6(6n+1)_5 + 15(4n)_5 - 20(2n-1)_5] = \frac{2n+1}{5!}\,(416n^4 + 832n^3 + 584n^2 + 168n),$$

$$(2.12)\quad C_{0,6} = \frac{1}{5!}\,[(6n+1)_5 - 6(4n)_5 + 15(2n-1)_5] = \frac{(2n+1)}{5!}\,(1056n^4 + 2112n^3 + 1704n^2 + 648n + 120).$$

For $m = 7, 8$, we have from (2.2) after simplification

$$(2.13)\quad C_{6n+3,7} = \frac{1}{6!}\,[(13n+4)_6 - 7(11n+3)_6 + 21(9n+2)_6 - 35(7n+1)_6 + 35(5n)_6 - 21(3n-1)_6 + 7(n-2)_6] = \frac{1}{6!}\,(n^6 + 3n^5 + 5n^4 - 15n^3 + 4n^2 + 12n),$$

$$(2.14)\quad C_{4n+2,7} = \frac{1}{6!}\,[(11n+3)_6 - 7(9n+2)_6 + 21(7n+1)_6 - 35(5n)_6 + 35(3n-1)_6 - 21(n-2)_6] = \frac{1}{6!}\,(722n^6 + 2166n^5 + 2060n^4 + 510n^3 - 262n^2 - 156n).$$

$$(2.15)\quad C_{2n+1,7} = \frac{1}{6!}\,[(9n+2)_6 - 7(7n+1)_6 + 21(5n)_6 - 35(3n-1)_6 + 35(n-2)_6] = \frac{1}{6!}\,[10543n^6 + 31629n^5 + 38845n^4 + 24975n^3 + 8572n^2 + 1356n],$$

$$(2.16)\quad C_{0,7} = \frac{1}{6!}\,[(7n+1)_6 - 7(5n)_6 + 21(3n-1)_6 - 35(n-2)_6] = \frac{1}{6!}\,[23548n^6 + 70644n^5 + 91000n^4 + 64260n^3 + 26572n^2 + 6216n + 720],$$

$$(2.17)\quad C_{6n+3,8} = \frac{1}{7!}\,[(14n+4)_7 - 8(12n+3)_7 + 28(10n+2)_7 - 56(8n+1)_7 + 70(6n)_7 - 56(4n-1)_7 + 28(2n-2)_7] = \frac{2n+1}{7!}\,[64n^6 + 192n^5 + 16n^4 - 288n^3 - 80n^2 + 96n],$$

$$(2.18)\quad C_{4n+2,8} = \frac{1}{7!}\,[(12n+3)_7 - 8(10n+2)_7 + 28(8n+1)_7 - 56(6n)_7 + 70(4n-1)_7 - 56(2n-2)_7] = \frac{(2n+1)}{7!}\,[7680n^6 + 23040n^5 + 23424n^4 + 8448n^3 - 864n^2 - 1248n],$$

$$(2.19)\quad C_{2n+1,8} = \frac{1}{7!}\,[(10n+2)_7 - 8(8n+1)_7 + 28(6n)_7 - 56(4n-1)^7 + 70(2n-2)^7] = \frac{(2n+1)}{7!}\,[76224n^6 + 228672n^5 + 282480n^4 + 183840n^3 + 64656n^2 + 10848n],$$

$$(2.20)\quad C_{0,8} = \frac{1}{7!}\,[(8n+1)_7 - 8(6n)_7 + 28(4n-1)_7 - 56(2n-2)_7] = \frac{(2n+1)}{7!}\,[154624n^6 + 463872n^5 + 597760n^4 + 422400n^3 + 174976n^2 + 41088n + 5040].$$

Thus we have completely found the sums

$$\sum_{k=-n}^{n} l_{kn}^{m}(t), \qquad m = 5,6,7,8.$$

After making the substitutions of these sums on the left hand side of (1.7) and performing cumbersome calculations we can verify the identity (1.7).

3. Proof of the theorem. We shall prove the theorem only for $r = 1$. For $r = 0$, the proof follows on the same pattern. We first notice that on account of (1.5) and the identity (1.7) we have

$$\sum_{k=0}^{n} q_{kn}(x) - 1 = \sum_{k=-n}^{n} p_{kn}(t) - 1 =$$

$$= \frac{14(n+2)(n+1)n(n-1)}{43(2n+1)^6}[10 - 15\cos(2n+1)t + 6\cos(4n+2)t - \cos(6n+3)t],$$

which gives

$$\left|\sum_{k=0}^{n} q_{kn}(x) - 1\right| \leqslant \frac{1}{n^2}. \tag{3.1}$$

Now owing to (1.4), we can write

$$\begin{aligned} Q_{n1}(f, x) - f(x) &= \left[\frac{1+x}{2}\{f(1) - f(x)\} + \frac{1-x}{2}\{f(-1) - f(x)\}\right]\cdot\left[1 - \sum_{k=0}^{n} q_{kn}(x)\right] - \\ &\quad - \sum_{k=0}^{n}[f(x) - f(x_{kn}) - (x - x_{kn})f'(x_{kn})]\cdot q_{kn}(x) = \\ &= \left[\frac{1+x}{2}\{f(1) - f(x) - (1-x)f'(x)\} + \frac{1-x}{2}\{f(-1) - f(x) + (1+x)f'(x)\}\right]\cdot\left[1 - \sum_{k=0}^{n} q_{kn}(x)\right] - \\ &\quad - \sum_{k=0}^{n}[f(x) - f(x_{kn}) - (x - x_{kn})f'(x_{kn})]\cdot q_{kn}(x) = T_1 + T_2. \end{aligned} \tag{3.2}$$

Making use of the well-known relation

$$f(u) - f(v) - (u - v)f'(v) = O(|u - v|)\omega_{f'}(|u - v|), \tag{3.3}$$

we have

$$\begin{aligned} |T_1| &\leqslant C_2\frac{1-x^2}{2}[\omega_{f'}(1-x) + \omega_{f'}(1+x)]\left|1 - \sum_{k=0}^{n} q_{kn}(x)\right| \leqslant \\ &\leqslant \frac{2}{n^2}C_2(1-x^2)\omega_{f'}(1) \leqslant 4C_2\frac{\sqrt{1-x^2}}{n}\cdot\omega_{f'}\left(\frac{\sqrt{1-x^2}}{n}\right) \leqslant \\ &\leqslant 4C_2\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)\omega_{f'}\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right). \end{aligned} \tag{3.4}$$

Further replacing x by $\cos t$ and x_{kn} by $\cos t_{kn}$ and using (1.5) we can write

$$T_2 = \sum_{k=-n}^{n}[f(\cos t) - f(\cos t_{kn}) - (\cos t - \cos t_{kn})f'(\cos t_{kn})]\cdot p_{kn}(t). \tag{3.5}$$

Denote by t_j the nearest node to t i.e. let

$$|t - t_j| \leqslant \frac{\pi}{2n+1}. \tag{3.6}$$

Since the functions involved in the expression for T_2 do not change if we increase or decrease the numbers k in multiples of $(2n+1)$, we have

$$T_2 = \sum_{k=j-n}^{j+n}[f(\cos t) - f(\cos t_{kn}) - (\cos t - \cos t_{kn})f'(\cos t_{kn})]\cdot p_{kn}(t). \tag{3.7}$$

In the following we estimate $|T_2|$.

On account of (1.3) we write

$$T_2 = T_2^{(1)} + T_2^{(2)}$$

where

$$T_2^{(1)} = \frac{1008}{43}\sum_{k=j-n}^{j+n}[f(\cos t) - f(\cos t_{kn}) - (\cos t - \cos t_{kn})f'(\cos t_{kn})]l_{kn}^{8}(t) \tag{3.8}$$

and

$$\begin{aligned} T_2^{(2)} &= \frac{1}{43}\sum_{k=j-n}^{j+n}[f(\cos t) - f(\cos t_{kn}) - \\ &\quad - (\cos t - \cos t_{kn})f'(\cos t_{kn})][-1820 l_{kn}^{6}(t) + 930 l_{kn}^{7}(t) - 105 l_{kn}^{8}(t)]. \end{aligned} \tag{3.9}$$

Now on account of (3.3) and the inequality

$$|l_{kn}(t)| \leqslant 1,$$

we have

$$|T_2^{(2)}| \leqslant 68\sum_{k=j-n}^{j+n}|\cos t - \cos t_{kn}|\,\omega_{f'}(|\cos t - \cos t_{kn}|)l_{kn}^{6}(t). \tag{3.10}$$

Since

$$|\cos t - \cos t_{kn}| \leqslant 2\sin t\sin\frac{|t - t_{kn}|}{2} + 2\sin^2\frac{t - t_{kn}}{2} \tag{3.11}$$

and

$$\begin{aligned} &\omega_{f'}(|\cos t - \cos t_{kn}|) \leqslant \\ &\leqslant 2n^2\sin^2\frac{t - t_{kn}}{2}\,\omega_{f'}(1/n^2) + 2n\sin\frac{|t - t_{kn}|}{2}\,\omega_{f'}\left(\frac{\sin t}{n}\right), \end{aligned} \tag{3.12}$$

we have

$$(3.13)\qquad |\cos t - \cos t_{kn}|\omega_{f'}(|\cos t - \cos t_{kn}|) \leqslant$$

$$\leqslant 4\omega_{f'}(1/n^2)\left[n^2 \sin t \sin^3 \frac{|t-t_{kn}|}{2} + n^2 \sin^4 \frac{t-t_{kn}}{2}\right] +$$

$$+ 4\left[n \sin t \sin^2 \frac{t-t_{kn}}{2} + n \sin^3 \frac{|t-t_{kn}|}{2}\right]\omega_{f'}\left(\frac{\sin t}{n}\right).$$

Hence from (3.10) we have

$$(3.14)\qquad |T_2^{(2)}| \leqslant 272n^2\omega_{f'}(1/n^2)\sum_{k=j-n}^{j+n}\left[\sin t \sin^3\frac{|t-t_{kn}|}{2} + \sin^4\frac{t-t_{kn}}{2}\right] l_{kn}^6(t) +$$

$$+ 272n\,\omega_{f'}\left(\frac{\sin t}{n}\right)\sum_{k=j-n}^{j+n}\left[\sin t \sin^2\frac{t-t_{kn}}{2} + \sin^3\frac{|t-t_{kn}|}{2}\right] l_{kn}^6(t).$$

Now we can easily show (see for example [4, lemma 3]) that

$$(3.15)\qquad \sum_{k=j-n}^{j+n} \sin^p \frac{|t-t_{kn}|}{2} l_{kn}^6(t) \leqslant \frac{1}{(2n+1)^p}, \quad p = 2, 3, 4.$$

Thus from (3.11) and (3.12) we have

$$(3.16)\qquad |T_2^{(2)}| \leqslant 272\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)\omega_{f'}\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right).$$

For the estimation of $|T_2^{(1)}|$ we shall use the idea of VERTESI-KIS [3] for estimating such sums pairwise. We write

(3.17)

$$T_2^{(1)} = \frac{1008}{43}\left[f(\cos t) - f(\cos t_{jn}) - (\cos t - \cos t_{jn})f'(\cos t_{jn})\right]\cdot l_{jn}^5(t) +$$

$$+ \frac{1008}{43}\sum_{k=j+1}^{j+n}\left[f(\cos t) - f(\cos t_{kn}) - (\cos t - \cos t_{kn})f'(\cos t_{kn})\right]\cdot l_{kn}^5(t) +$$

$$+ \frac{1008}{43}\sum_{k=j-n}^{j-1}\left[f(\cos t) - f(\cos t_{kn}) - (\cos t - \cos t_{kn})f'(\cos t_{kn})\right] l_{kn}^5(t).$$

We denote these parts respectively by S_1, S_2, S_3 and show that each has the order

$$O\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)\omega_{f'}\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right).$$

On account of (3.3) and (3.11), we have

$$|S_1| \leqslant 96n^2\omega_{f'}(1/n^2)\left[\sin t \sin^3\frac{|t-t_{jn}|}{2} + \sin^4\frac{t-t_{jn}}{2}\right] l_{jn}^5(t) +$$

$$+ 96n\omega_{f'}\left(\frac{\sin t}{n}\right)\left[\sin t \sin^2\frac{t-t_{jn}}{2} + \sin^3\frac{|t-t_{jn}|}{2}\right] l_{jn}^5(t).$$

Now

$$\left|\sin^p \frac{|t-t_{jn}|}{2} l_{jn}^5(t)\right| = \left|\frac{\sin^5 \frac{2n+1}{2}(t-t_{jn})}{(2n+1)^5 \sin^{5-p}\frac{t-t_{jn}}{2}}\right| \leqslant \frac{1}{(2n+1)^p}, \qquad p = 2, 3, 4,$$

hence

$$(3.18)\qquad |S_1| \leqslant 24\left(\frac{\sin t}{n} + \frac{1}{n^2}\right)\omega_{f'}\left(\frac{\sin t}{n} + \frac{1}{n^2}\right).$$

The estimations of the sums S_2 and S_3 are similar, so we estimate only the sum

(3.19)

$$|S_2| = \left|\frac{1008}{43}\sum_{j-n}^{j+n}[f(\cos t) - f(\cos t_{kn}) - (\cos t - \cos t_{kn})f'(\cos t_{kn})[\,l_{kn}^5(t)\right|$$

We consider the estimates of the pairs

$$[f(\cos t) - f(\cos t_{kn}) - (\cos t - \cos t_{kn})f'(\cos t_{kn})]\,l_{kn}^5(t) +$$

$$+ [f(\cos t) - f(\cos t_{k+1,n}) - (\cos t - \cos t_{k+1,n})f'(\cos t_{k+1,n})]\cdot l_{k+1,n}^5(t).$$

If n is odd then apart from such pairs which are $\frac{n-1}{2}$ in numbers, there will be an extra term

$$[f(\cos t) - f(\cos t_{j+n,n}) - (\cos t - \cos t_{j+n,n})f'(\cos t_{j+n,n})]\cdot l_{j+n,n}^5(t).$$

This term is estimated in the same way as the terms in $|S_1|$ and give the desired order.

So we assume n to be even and estimate each pair as follows: Let $k = j+1, j+2, \ldots, j+n-1$. We write

$$|[f(\cos t) - f(\cos t_{kn}) - (\cos t - \cos t_{kn})f'(\cos t_{kn})]l_{kn}^5(t) +$$
$$+ [f(\cos t) - f(\cos t_{k+1,n}) - (\cos t - \cos t_{k+1,n})f'(\cos t_{k+1,n})]\cdot l_{k+1,n}^5(t)| =$$
$$= |[f(\cos t) - f(\cos t_{kn}) - (\cos t - \cos t_{kn})f'(\cos t_{kn})]\cdot[l_{kn}^5 + l_{k+1,n}^5(t)] +$$
$$+ (\cos t - \cos t_{k+1,n})(f'\cos t_{kn}) - f'(\cos t_{k+1,n}))l_{k+1}^5(t) + [f(\cos t_{kn}) -$$
$$- f(\cos t_{k+1,n}) - (\cos t_{kn} - \cos t_{k+1,n})f'(\cos t_{k+1,n})]l_{k+1,n}^5(t)| \leqslant$$
$$\leqslant C_3|\cos t - \cos t_{kn}|\,\omega_{f'}(|\cos t - \cos t_{kn}|\,|l_{kn}^5(t) + l_{k+1,n}^5(t)| +$$
$$+ \cos t - \cos t_{k+1,n}|\omega_{f'}(|\cos t_{kn} - \cos t_{k+1,n}|)\cdot|l_{k+1,n}^5(t)| +$$
$$+ |\cos t_{kn} - \cos t_{k+1,n}|\cdot\omega_{f'}(|\cos t_{kn} - \cos t_{k+1,n})|l_{k+1,n}^5(t)| =$$
$$= S_2^{(1)} + S_2^{(2)} + S_2^{(3)}.$$

We first show

$$|l_{kn}^5(t) + l_{k+1,n}^5(t)| \leqslant \frac{5\pi}{\sigma_k^6} \tag{3.20}$$

and

$$\omega_{f'}(|\cos t_{kn} - \cos t_{k+1,n}|) \leqslant$$

$$\leqslant 4\omega_{f'}\left(\frac{\sin t}{n}\right) + \left(2n\pi \sin\frac{|t - t_{k+1,n}|}{2} + 1\right)\omega_{f'}\left(\frac{1}{n^2}\right), \tag{3.21}$$

where

$$\sigma_k = (2n+1)\sin\frac{(t - t_{kn})}{2}, \quad j < k < j+n.$$

We have

$$|l_{kn}^5(t) + l_{k+1,n}^5(t)| = \left|\frac{(-1)^{5k}\sin^5\frac{2n+1}{2}t}{(2n+1)^5}\cdot\left[\frac{1}{\sin^5\frac{t-t_{kn}}{2}} - \frac{1}{\sin^5\frac{t-t_{k+1,n}}{2}}\right]\right| =$$

$$= \left|(-1)^{5k}(2n+1)\sin^5\frac{2n+1}{2}t\left(\sin\frac{t-t_{k+1,n}}{2} - \sin\frac{t-t_{kn}}{2}\right)\times\right.$$
$$\left.\times\left\{\frac{1}{\sigma_{kn}^5\sigma_{k+1,n}} + \frac{1}{\sigma_{kn}^4\sigma_{k+1,n}^2} + \frac{1}{\sigma_{kn}^3\sigma_{k+1,n}^3} + \frac{1}{\sigma_{kn}^2}\cdot\frac{1}{\sigma_{k+1,n}^4} + \frac{1}{\sigma_{kn}\sigma_{k+1,n}^5}\right\}\right| \leqslant \frac{5\pi}{\sigma_k^6}$$

To show (3.19), we observe that

$$|\cos t_{kn} - \cos t_{k+1,n}| = \left|2\sin\frac{\pi}{2n+1}\sin\frac{2k+1}{2n+1}\right| \leqslant \frac{2\pi}{2n+1}|\sin t_{k+1,n}| \leqslant$$
$$\leqslant \frac{2\pi}{2n+1}\sin t + \frac{4\pi}{2n+1}\cdot\sin\frac{|t - t_{k+1,n}|}{2},$$

from which (3.19) is obvious after using the properties of modulus of continuity. Now on account of (3.11), (3.12) and (3.20)

$$|S_2^{(1)}| \leqslant 5\pi\left[\frac{\sin t}{(2n+1)^4\sin^3\frac{|t-t_{kn}|}{2}} + \frac{1}{(2n+1)^4\sin^2\frac{t-t_{kn}}{2}}\right]\omega_{f'}\left(\frac{1}{n^2}\right) + \tag{3.23}$$
$$+ 5\pi\left[\frac{\sin t}{(2n+1)^3\sin^4\frac{t-t_{kn}}{2}} + \frac{1}{(2n+1)^5\sin^3\frac{|t-t_{kn}|}{2}}\right]\omega_{f'}\left(\frac{\sin t}{n}\right).$$

From (3.11) and (3.21) we have

$$|S_2^{(2)}| \leqslant \frac{8}{(2n+1)^5}\omega_{f'}\left(\frac{\sin t}{n}\right)\left[\frac{\sin t}{\sin^4\frac{t-t_{k+1,n}}{2}} + \frac{1}{\sin^3\frac{|t-t_{k+1,n}|}{2}}\right] + \tag{3.24}$$
$$+ \frac{2}{(2n+1)^5}\omega_{f'}\left(\frac{1}{n^2}\right)\left[\frac{\sin t}{\sin^4\frac{t-t_{k+1,n}}{2}} + \frac{1}{\sin^3\frac{|t-t_{k+1,n}|}{2}} +\right.$$
$$\left. + \frac{2n\pi}{\sin^3\frac{|t-t_{k+1,n}|}{2}} + \frac{2n\pi}{\sin^2\frac{t-t_{k+1,n}}{2}}\right].$$

Similarly from (3.21) and (3.22) we have

$$|S_2^{(3)}| \leqslant \frac{8\pi}{(2n+1)^6}\omega_{f'}\left(\frac{\sin t}{n}\right)\left[\frac{\sin t}{\sin^5\frac{|t-t_{k+1,n}|}{2}} + \frac{2}{\sin^4\frac{t-t_{k+1,n}}{2}}\right] + \tag{3.25}$$
$$+ \frac{2\pi}{(2n+1)^6}\omega_{f'}\left(\frac{1}{n^2}\right)\left[\frac{2n\pi\sin t}{\sin^4\frac{t-t_{k+1,n}}{2}} + \frac{\sin t}{\sin^5\frac{|t-t_{k+1,n}|}{2}} +\right.$$
$$\left. + \frac{4n\pi}{\sin^3\frac{|t-t_{k+1,n}|}{2}} + \frac{2}{\sin^4\frac{t-t_{k+1,n}}{2}}\right].$$

Hence

$$|S_2| \leqslant \frac{1008}{43} \sum_{k=j+1,j+3,\ldots,j+n-1} \omega_{f'}\left(\frac{1}{n^2}\right)\left[\frac{5\pi \sin t}{(2n+1)|\sigma^{8}_{kn}|} + \right.$$

$$+ \frac{5\pi}{(2n+1)^2\sigma^2_{kn}} + \frac{2 \sin t}{(2n+1)} \cdot \frac{1}{\sigma^4_{k+1,n}} + \frac{2}{(2n+1)^2|\sigma^3_{k+1,n}|} +$$

$$+ \frac{4n\,\pi \sin t}{(2n+1)^2|\sigma_{k+1,n}|^3} + \frac{4n\pi}{(2n+1)^3\sigma^2_{k+1,n}} + \frac{4n\,\pi^2 \sin t}{(2n+1)^2\sigma^2_{k+1,n}} + \frac{2\pi \sin t}{(2n+1)|\sigma_{k+1,n}|^5} +$$

$$\left. + \frac{4n\pi^2}{(2n+1)^3\sigma^3_{k+1,n}} + \frac{4\pi}{(2n+1)^2\sigma^4_{k+1,n}}\right] + \omega_{f'}\left(\frac{\sin t}{n}\right)\left[\frac{5\pi \sin t}{(2n+1)\sigma^4_{kn}} + \frac{1}{(2n+1)^2|\sigma^3_{kn}|} + \right.$$

$$+ \frac{8 \sin t}{(2n+1)\sigma^4_{k+1,n}} + \frac{8}{(2n+1)^2|\sigma_{k+1,n}|^3} + \frac{8\pi \sin t}{(2n+1)|\sigma^5_{k+1,n}|} + \frac{16\pi}{(2n+1)^2\sigma^4_{k+1}} \leqslant$$

$$\leqslant 24\omega_{f'}\frac{1}{n^2} \sum_{k=j+1,j+3,\ldots,j+n-1}\left[10\frac{\sin t}{n}\frac{1}{|\sigma^3_{kn}|} + \frac{5}{n^2\,\sigma^2_{kn}} + \frac{\sin t}{n\,\sigma^4_{k+1,n}} + \frac{1}{n^2|\sigma^3_{k+1,n}|} + \right.$$

$$\left. + \frac{4 \sin t}{n|\sigma^3_{k+1,n}|} + \frac{2}{n^2\sigma^2_{k+1,n}} + \frac{16 \sin t}{n\,\sigma^2_{k+1}} + \frac{4 \sin t}{n|\sigma^5_{k+1,n}|} + \frac{8}{n^2|\sigma^3_{k+1,n}|} + \frac{4}{n^2\sigma^4_{k+1,n}}\right] +$$

$$+ 24\,\omega_{f'}\left(\frac{\sin t}{n}\right) \sum_{k=j=1,j+3,\ldots,j+n-1}\left[\frac{10 \sin t}{n\sigma^4_{kn}} + \frac{1}{n^2|\sigma^2_{kn}|} + \frac{4 \sin t}{n\sigma^4_{k+1,n}} + \frac{2}{n^2|\sigma^3_{k+1,n}|} + \right.$$

$$\left. + \frac{16 \sin t}{n|\sigma^5_{k+1,n}|} + \frac{16}{n^2\sigma^4_{k+1,n}}\right] \leqslant 24\frac{\sin t}{n}\omega_{f'}\left(\frac{1}{n^2}\right)\sum_{k=1}^{\infty}\frac{35}{\sigma^2_{kn}} + \frac{24}{n^2}\omega_{f'}\left(\frac{1}{n^2}\right)\sum_{k=1}^{\infty}\frac{20}{\sigma^2_{kn}} +$$

$$+ 24\frac{\sin t}{n}\omega_{f'}\left(\frac{\sin t}{n}\right)\sum_{k=1}^{\infty}\frac{30}{\sigma^2_{kn}} + 24\frac{1}{n^2}\,\omega_{f'}\left(\frac{\sin t}{n}\right)\sum_{k=1}^{\infty}\frac{19}{\sigma^2_{kn}} \leqslant 840\,\omega_{f'}\left(\frac{1}{n^2}\right)\left[\frac{\sin t}{n} + \frac{1}{n^2}\right] \times$$

$$\times \sum_{k=1}^{\infty}\frac{1}{\sigma^2_{kn}} + 720\,\omega_{f'}\left(\frac{\sin t}{n}\right)\left[\frac{\sin t}{n} + \frac{1}{n^2}\right] \cdot \sum_{k=1}^{\infty}\frac{1}{\sigma^2_{kn}} \leqslant 840\left[\frac{\sin t}{n} + \frac{1}{n^2}\right] \times$$

$$\times \left[\omega_{f'}\left(\frac{\sin t}{n} + \frac{1}{n^2}\right)\right] \cdot \sum_{k=1}^{\infty}\frac{1}{\sigma^2_{kn}} \leqslant 2520\left[\frac{\sin t}{n} + \frac{1}{n^2}\right]\omega_{f'}\left(\frac{\sin t}{n} + \frac{1}{n^2}\right).$$

In this way we have shown

(3.26) $$|T_2^{(1)}| \leqslant C_4\left[\frac{\sin t}{n} + \frac{1}{n^2}\right]\omega_{f'}\left(\frac{\sin t}{n} + \frac{1}{n^2}\right).$$

Thus from (3.7), (3.16) and (3.26) we have

(3.27) $$|T_2| \leqslant C_5\left(\frac{\sin t}{n} + \frac{1}{n^2}\right)\omega_{f'}\left(\frac{\sin t}{n} + \frac{1}{n^2}\right).$$

Hence from (3.2), (3.4) and (3.27) we have obtained the desired result and this completes the proof of our theorem.

REFERENCES

[1] Gopengauz, I. E., *On a theorem of A. F. Timon on the approximation of functions by polynomials on finite segment.* Mat. Zame. 2, 163—172 (1967) (Russian).

[2] Jackson, D., *Theory of approximation.* Amer. Math., Soc. Coll. Pub. vol. XI, New York, 1930.

[3] Kis, O., Vertesi, P., *On a new interpolation process* (Russian). Annals Univ. Sci. Budapest, **X**, 117—128 (1967).

[4] Saxena, R. B. & Srivastava, K. B., *On interpolation operators I* (A proof of Jackson's theorem for differentiable functions). L'analyse numérique, et la théorie de l'approximation, **7**, *2*, 211—223 (1978).

[5] Telyakovskii, S. A., *Two theorems on the approximation of functions by algebraic polynomials.* Math. Sbornik, **70**, *2*, 252—265 (Russian).

[6] Timan, A. F., *A strengthening of Jackson's theorem on the best approximation of continuous functions by polynomials a finite segment of the real axis.* Dokl Akad. Nauk. SSSR, **78**(1951), 17—20 (Russian); MR 12, 823.

[7] Tureckii, A. H. *On certain extremal problems in the theory of interpolation.* A collection of modern research problems in the constructive theory, of functions, Baku Publishor, AN ASSR (1965) 220—232 (Russian).

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