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ON THE LEVEL-UPCROSSINGS OF STOCHASTIC PROCESSES

by

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ABSTRACT. In this paper the probability of n upcrossings in an interval is obtained. Then the probability of at least n upcrossings in an interval is obtained in terms of certain multiple integrals. The distribution function of the interval between an arbitrary upcrossing and the next one is considered in detail and a formula for its n-th moment as a double integral involving the first occurrence density is obtained for a general stochastic process. Finally for an ergodic process the n-th moment is expressed in terms of the $(n-2)$ nd moment of the probability of no crossings in the interval $(0, t)$.

1. Introduction

The problem of finding the probabilities of some random variables associated with the upcrossings of a fixed level by random processes is of profound practical importance (BLAKE and LINDSEY [2]). For example the analysis of structural systems subjected to random loading (KAMEDA [5]), communication theory and guidence systems are a few of the many areas of such application.

2. Some Fundamental Relationships

Let $x(t)$ be a continuous random variable. Here we shall consider some random variables associated with the crossings of $x(t)$ with the level $|x(t)| = L$ in its duration T. Let $E_n(t)$ be the event that $|x(t)|$ exceeds the level L $(n-1)$ times in the interval $(0, t)$ and the n-th upcrossing of the level L occurs at time t in the horizontal window sense of KAC and SLEPIAN [4]. Define the probability densities $p_n(t)$ and $\tilde{p}_n(t)$ as follows:

$$p_n(t)dt = P[E_n(t) \; : |x(0)| \leqslant L]$$

and

(2.1) $$\tilde{p}_n(t)dt = P[E_n(t) : |x(0)| > L].$$

For $n = 1$, $p_1(t)$ is the first occurrence density considered by RICE and BEER [7].

The probability of at least n upcrossings of the level L in the interval $(0, T)$ conditional to $|x(0)| \leqslant L$ is thus given by $\int_0^T p_n(t)dt$. Similarly $\int_0^T \tilde{p}_n(t)\,dt$ is the corresponding conditional probability given $|x(0)| > L$. Thus the probability of at least n upcrossings in the interval $(0, T)$ is given by

(2.2) $$a_0 \int_0^T p_n(t)dt + \tilde{a}_0 \int_0^T \tilde{p}_n(t)dt,$$

where $a_0 = P[|x(0)| \leqslant L]$ and $\tilde{a}_0 = P[|x(0)| > L]$.
If the event $[|x(0)| < L]$ is considered to be the first upcrossing of the level L, the probability of exactly n upcrossings in $(0, t)$ is represented by

(2.3) $$U_n(t) = a_0\left\{\int_0^t \tilde{p}_n(\tau)d\tau - \int_0^t p_{n+1}(\tau)d\tau\right\} + \tilde{a}_0\left\{\int_0^t \tilde{p}_{n-1}(\tau)d\tau - \int_0^t p_n(\tau)d\tau\right\}; \quad n \geqslant 1.$$

For $n = 0$

(2.4) $$U_0(t) = a_0\left\{1 - \int_0^t p_1(\tau)d\tau\right\}.$$

Notice that equation (2.3) gives for $n = 1$

$$U_1(t) = a_0\left\{\int_0^t p_1(\tau)d\tau - \int_0^t p_2(\tau)d\tau\right\} + a_0\left\{1 - \int_0^t \tilde{p}_1(\tau)d\tau\right\},$$

by the assumption that $[|x(0)| > L]$ is the first upcrossing.

Equation (2.3) is a generalization of a formula given by RICE and BEER [7] for the probability of failure of a mechanical system subjected to a random loading.

The probability of at most n upcrossings in $(0, t)$ is given by

$$V_n(t) = \sum_{k=0}^{n} U_k(t)$$

which by substitution from equation (2.3) would give

(2.5) $$V_n(t) = \begin{cases} a_0\left\{1 - \int_0^t p_1(\tau)d\tau\right\}; & n = 0 \\ 1 - \left\{a_0 \int_0^t p_{n+1}(\tau)d\tau + \tilde{a}_0 \int_0^t \tilde{p}_n(\tau)d\tau\right\}; & n \geqslant 1 \end{cases}$$

3. An Inclusion-Exclusion Formula for $p_n(t)$ and $\tilde{p}_n(t)$

Here we shall use a technique developed by BARTLETT [1] to obtain a representation for $p_n(t)$ and $\tilde{p}_n(t)$ in terms of certain multiple integrals.

Divide the interval $(0, t)$ into m equal subintervals $\Delta_1, \Delta_2, \ldots, \Delta_m$ and define the two events e_i and $\bar{e}_i$ to denote an upcrossing or no upcrossing of the level L in Δ_i respectively.

Take Δ_i, $i = 1, 2, \ldots, m$ so small as only one crossing could take place, if any, in Δ_i. Hence if $Q_n(m)$ represents the event that the n-th upcrossing of the level $|x(t)| = L$ takes place in the interval Δ_m, $m \geqslant n$, then $Q_n(m)$ can be written as the Union of $\frac{(m-1)n}{(m-n)!}$ mutually exclusive events of the from

$$[\bar{e}_1 \cap \bar{e}_2 \cap \ldots \cap \bar{e}_{i_1-1} \cap e_{i_1} \cup \bar{e}_{i_1+1} \cap \ldots \cap \bar{e}_{i_2-1} \cap e_{i_2} \cup \bar{e}_{i_2+1} \cap \ldots \cap \bar{e}_{i_{n-1}-1} \cap e_{i_{n-1}} \cap \bar{e}_{i_{n-1}+1} \cap \ldots \cap \bar{e}_{m-1} \cap e_m].$$

Thus

(3.1) $$P[Q_n(m)] = P\Big[\bigcup_{i=1}^{m-n+1} \bigcup_{i_2=i_1+1}^{m-n+2} \ldots \bigcup_{i_{n-1}=i_{n-2}+1}^{m-1} (\bar{e}_1 \cap \bar{e}_2 \cap \ldots \cap \bar{e}_{i_1-1} \cap e_{i_1} \cap \bar{e}_{i_1+1} \cap \ldots \cap \bar{e}_{i_2-1} \cap e_{i_2} \cap \bar{e}_{i_2+1} \cap \ldots \cap \bar{e}_{i_{n-1}-1} \cup e_{i_{n-1}} \cap \bar{e}_{i_{n-1}+1} \cap \ldots \cap \bar{e}_{m-1} \cap e_m)\Big]$$

replacing $\bar{e}_i$ by $1 - e_i$ and after simple manipulations the last formula is written in the form

(3.2)

$$P[Q_n(m)] = \sum_{i_1=1}^{m-n+1} \sum_{i_2=i_1+1}^{m-n+2} \cdots \sum_{i_{n-1}=i_{n-2}+1}^{m-1} \left\{ P\left[\left(\bigcap_{k=1}^{n-1} e_{i_k}\right) \cap e_m\right] - \sum_{i_n=1}^{m-1} P\left[\left(\bigcap_{k=1}^{n} e_{i_k}\right) \cap e_m\right] + \sum_{i_n=1}^{m-2} \sum_{i_{n+1}=i_{n+1}}^{m-1} P\left[\left(\bigcap_{k=1}^{n+1} e_{i_k}\right) \cap e_m\right] - \ldots \right\}$$

Taking the limit as $\Delta_i \to 0$, $p_n(t)$ is written in the form

$$p_n(t) = \lim_{\Delta \to 0} P[Q_n(m) : |x(0)| \leqslant L] =$$

(3.3)

$$= \int_0^t dt_1 \int_{t_1}^t dt_2 \ldots \int_{t_{n-2}}^t \Big\{ f_s(t_1, t_2, \ldots, t_{n-1}, t) - \sum_{k=0}^{\infty} (-1)^k \int_0^t dt_n \int_{t_n}^t dt_{n-1} \ldots \int_{t_{n+k-1}}^t f_s(t_1, t_2, \ldots, t_{n+k}) dt_{n+k} \Big\}$$

where

(3.4)

$$f_s(t_1, t_2, \ldots, t_k) dt_1, dt_2 \ldots dt_k =$$

$$= P\left[\bigcap_{i=1}^{k} \{|x(t_i)| \leqslant L \cap |x(t_i + dt_i) > L = |x(0)| \leqslant L\right],$$

i.e.

$f_s(t_1, t_2, \ldots, t_k) dt_1 dt_2 \ldots dt_k$ is the conditional probability of upcrossings in $(t_1, t_1 + dt_1)$, $(t_2, t_2 + dt_2)$, $\ldots$, $(t_k, t_k + dt_k)$ given that $|x(0)| \leqslant L$.

The independent variables in $f_s(t_1, t_2, \ldots, t_k)$ can be interchanged arbitrarily provided that $x(t)$ has the same distribution throughout its time duration. Thus equation (3.3) can be written in the form

(3.5)

$$p_n(t) = \sum_{i=1}^{\infty} (-1)^{i-1} \binom{n+i-2}{n-1} \int_0^t dt_1 \int_{t_1}^t dt_2 \ldots \int_{t_{n+i-3}}^t f_s(t_1, t_2, \ldots, t_{n+i-2}, t) dt_{n+i-2}.$$

A similar expression for $\tilde{p}_n(t)$ can be obtained where f_s is replaced by $\tilde{f}_s$ and $\tilde{f}_s(t_1, t_2, \ldots, t_k) dt_1 dt_2 \ldots dt_k$ is the conditional probability of upcrossings in $(t_1, t_1 + dt_1)$, $(t_2, t_2 + dt_2)$, $\ldots$, $(t_k, t_k + dt_k)$ given that $|x(0)| \leqslant L$.

Using equation (3.5), the conditional probability of at least n upcrossings in the interval $(0, t)$ given that $|x(0)| \leqslant L$ is represented by,

(3.6)

$$\int_0^t p_n(\tau) d\tau = \sum_{i=1}^{\infty} (-1)^{i-1} \binom{n+i-2}{n-1} \times \int_0^t d\tau \int_0^\tau dt_1 \int_{t_1}^\tau dt_2 \ldots \int_{t_{n+i-3}}^\tau f_s(t_1, t_2, \ldots, t_{n+i-2}, \tau) dt_{n+i-2} = \sum_{i=1}^{\infty} (-1)^{i-1} \binom{n+i-2}{n-1} \times$$

$$\times \int_0^t dt_1 \int_{t_1}^t dt_2 \ldots \int_{t_{n+i-3}}^t dt_{n+i-2} \int_{t_{n+i-2}}^t f_s(t_1, t_2, \ldots, t_{n+i-2}, \tau) d\tau.$$

Similarly

(3.7)

$$\int_0^t \tilde{p}_n(\tau) d\tau = \sum_{i=1}^{\infty} (-1)^{i-1} \binom{n+i-2}{n-1} \times \int_0^t dt_1 \int_{t_1}^t dt_2 \ldots \int_{t_{n+i-3}}^t f_s(t_1, t_2, \ldots, t_{n+i-2}, \tau) d\tau$$

4. Intervals between Upcrossings

In this section we shall assume that $x(t)$ is a strictly stationary process and that the mean number of crossings of the level L, per unit time, is finite and nonzero. Hence, the upcrossings of the level L form a stationary stream of events which is also regular in the sense that the probability of two more upcrossings in time t is $0(t)$ as $t \to 0$. By Korolyook's theorem (KHINTCHINE [6],) the probability of at least one upcrossing in the interval $(0, t)$ is given by (CRAMER and LEADBETTER [3])

(4.1) $W(t) = Mt + O(t)$ as $t \to 0$.

Using equation (2.5), $w(t)$ can be written in the form

(4.2)
$$w(t) = \sum_{n=1}^{\infty} U_n = \sum_{1=n}^{\infty} \left[1 - a_0 \int_0^t p_{n+1}(\tau) d\tau - \tilde{a}_0 \int_0 \tilde{p}_n(\tau) d\right].$$

Let $U(t_1, t_2)$, $D(t_1, t_2)$ and $C(t_1, t_2)$ be the number of upcrossings, downcrossings and crossings of the level L be $x(t)$ in (t_1, t_2) respectively. Define, for $\tau > 0$, $k = 0, 1, 2, \ldots$

(4.3) $R_k(\tau, t) = P[U(-\tau, 0) \geqslant 1, \quad C(0, t) \leqslant k]$

and

(4.4) $$S_k(\tau, t) = P[U(-\tau, 0) \geq 1,\ x(0) > U,\ C(0, t) \leq k].$$

Thus $R_k(\tau, t) - S_k(\tau, t) \leq P[U(-\tau, 0) \geq 1, D(-\tau, 0) \geq] \leq P\ [C(-\tau, 0) \geq 2]$.

By the regularity of the stream of crossings, the last inequality gives

(4.5) $$R_k(\tau, t) - S_k(\tau, t) \leq O(\tau).$$

Now

$$S_k(\tau_1 + \tau_2, t) = P[U(-\tau_1, 0) \geq 1,\ x(0) > U,\ C(0, t) \leq k] +$$
$$+ P[U(-\tau_1 - \tau_2, -\tau_1) \geq 1,\quad U(-\tau_1, 0) = 0,\ x(0) > U,\ C(0, t) \leq k],$$

The event in the second term on the right implies that there is no upcrossing or downcrossin in $(-\tau, 0)$ and thus also $x(-\tau_1) \geq U$, i.e. it implies the event $[U(-\tau_1 - \tau_2, -\tau_1) \geq 1,\ x(-\tau_1) \geq U,\ C(0, t) \leq k]$

and by stationary this is the same as

$$[U(-\tau_2, 0) \geq 1,\ x(0) > U,\ C(0, t + \tau_1) \leq k].$$

Thus

$$S_k(\tau_1 + \tau_2, t) \leq S_k(\tau_1, t) + S_k(\tau_2, t + \tau_1) \leq S_k(\tau_1, t) + S_k(\tau_2, t).$$

Using a lemma proved by KHINTCHINE [6], since $S_k(\tau, t)$ is nondecreasing as τ increases, $S_k(\tau, t)/\tau$ converges to a limit as $\tau \to 0$. Thus, from equation (4.5), $k_k(\tau, t)/\tau$ also converges to a limit as $\tau \to 0$ and hence by (4.1) $R_k(\tau, t)/w(\tau)$ tends to a limit as $\tau \to 0$. This limit is finite, since $k_k(\tau, t)/w(\tau) \leq 1$, and we write

(4.6) $$Z_k(t) = \lim_{\tau \to 0} \frac{R_k(\tau, t)}{w(\tau)},$$

$Z_k(t)$ represents the conditional probability of no more than k crossings in the interval $(0, t)$, given that an upcrossing occured at $t = 0$.

Define now

(4.7) $$F_k(t) = 1 - Z_{k-1}(t),\ k = 1, 2, \ldots$$

$F_1(t)$ denotes the conditional probability that there is at least one crossing in the interval $(0, t)$, given an upcrossing at $t = 0$. The distribution function of the length of the interval between an arbitrary upcrossing and the next upcrossing is represented by $F_2(t)$

By definition, $U_n(t)$ is the probability of n upcrossings in $(0, t)$, i.e. $U_n(t) = P[U(0, t) = n]$.

To express $F_2(t)$ in terms of $U_0(t)$, we have for $\tau > 0$:

$$U_0(t) - U_0(t + \tau) = P[U(0, t) = 0] - P[U(-\tau, t) = 0] =$$
$$= P[U(-\tau, 0) \geq 1,\ U(0, t) = 0] =$$
$$= R_1(\tau, t) + O(\tau).$$

Thus by equations (4.1) and (4.6)

(4.8) $$\lim_{\tau \to 0} \frac{U_0(t + \tau) - U_0(t)}{\tau} = -MZ_1(t)$$

and hence by using equation (4.7), the right-hand derivative $D^+U_0(t)$ exists and satisfies

(4.9) $$F_2(t) = 1 + M^{-1}D^+U_0(t).$$

5. Moments of the Interval between Upcrossings

The mean of $F_2(t)$ is given by

(5.1) $$\int_0^\infty t dF_2(t) = \int_0^\infty [1 - F_2(t)]dt,$$

where both members may be infinite. Therefore

(5.2) $$\int_0^\infty t dF_2(t) = -\frac{1}{M}\int_0^\infty D^+U_0(t)dt.$$

Since $$|U_0(t + \tau) - U_0(t)| \leq w(|\tau|),$$

it follows that $U_0(t)$ is a continuous function of t and hence the mean of $F_2(t)$ is given by

$$\int_0^\infty t dF_2(t) = \frac{1}{M}[1 - U_0(\infty)].$$

By using relation (2.5), we have

$$\int_0^\infty t dF_2(t) = \frac{1}{M}\left[1 - a_0\left\{1 - \int_0^\infty p_1(t)dt\right\}\right].$$

For an ergodic process, $\int_0^\infty p_1(t)dt = 1$ and hence

$$\int_0^\infty t dF_2(t) = \frac{1}{M}$$

an intuitively reasonable result.

For higher moments of $F_2(t)$, it is straight forward to show that

$$\text{(5.4)} \quad \int_0^\infty t^n d\,F_2(t) = \frac{n(n-1)}{M} \int_0^\infty [U_0(t) - U_0(\infty)] t^{n-2} dt =$$

$$= \frac{n(n-1)}{M} \int_0^\infty \left[a_0 \left\{ 1 - \int_0^t p_1(\tau)d\tau \right\} - a_0 \left\{ 1 - \int_0^\infty p_1(\tau)d\tau \right\} \right] t^{n-2} dt =$$

$$= \frac{n(n-1)}{M} a_0 \int_0^\infty \int_0^\infty t^{n-2} p_1(\tau) d\tau dt.$$

Again for an ergodic process, the n th moment of $F_2(t)$ is given by

(5.5)

$$\int_0^\infty t^n dF_2(t) = \frac{n(n-1)a_0}{M} \int_0^\infty t^{n-2} \left[1 - \int_0^t p_1(\tau)d\tau \right] dt = \frac{n(n-1)}{M} \int_0^\infty t^{n-2} U_0(t) dt.$$

The relations (5.4) and (5.5) hold in the sense that both sides are either finite and equal or both infinite.

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