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ON CONVERGENT SEQUENCES OF SECOND ORDER WITH RESPECT TO A MAPPING

by

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Let x be a Banach space and let's consider the equation

(1) $$P(x) = \theta,$$

where $P : X \to X$ is a continous mapping and θ is the null element of X. The equation can be written as follows:

(1') $$x - \Phi(x) = \theta.$$

The sequence (x_n) $(n = 0, 1, 2, \ldots)$. which consists of points of X, is called convergent of second order with respect to the mapping P, if (x_n) is convergent and a number $\alpha > 0$ exists, which does not depend on n, so that

$$\|P(x_{n+1})\| \leqslant \alpha \|P(x\ )\|^2$$

for all $n = 0, 1, 2, \ldots$ [4].

In this paper we shall give sufficient conditions so that a sequence (x_n) of the space X is convergent of second order with respect to the mapping P and the limit of the sequence is a solution of equation (1).

In the followings we shall note by $[z_1, z_2; P]$ and $[z_1, z_2, z_3; P]$ respectively the divided differences of first and second order in the points z_1, z_2, z_3. We shall also use the notation $u_n = \Phi(x_n)$., $n = 0, 1, 2, \ldots$.

THEOREM. *Let's presume the sequence (x_n), the mapping P and the number $\alpha > 0$ are so that in the ball $S(x_0, \delta) = \{x : \|x - x_0\| < \delta\}$ the following conditions are satisfied :*

1° $\sup \{\|[z_1, z_2, z_3; P]\| : z_1, z_2\ z_3 \in S(x_0, \delta)\} \leqslant M < \infty$;

2° *There exists a positive integer A which does not depend on n so that*

$$\|P(u_n) + [x_n, u_n; P](x_{n+1} - u_n)\| \leqslant A\|P(x_n)\|^2$$

for all $n = 1, 2, \ldots$;

3° *A positive number B exists, which does not depend on n so that*

$$\|x_{n+1} - x_n\| \leqslant B\|P(x_n)\| \cdot \|x_{n+1} - u_n\| \leqslant B\|P(x_n)\|$$

for any $n = 0, 1, 2, \ldots$;

4° $h_0 = \|P(x_0)\|(A + MB^2) = \eta_0\gamma < 1, \quad \delta = \frac{h_0(1+B)}{\gamma(1-h_0)}$.

In these conditions the sequence (x_n) *has the following qualities:*

(i) (x_n) *is convergent of second order with respect to the mapping P and the limit* x^* *of the sequence is a solution of equation* (1);

(ii) $$x^* \in S(x_0, \delta);$$

(iii) $$\|x_{n+1} - x_n\| \leqslant \frac{Bh_0^{2^n}}{\gamma} \text{ for } n = 0, 1, 2, \ldots;$$

(iv) $$\|x^* - x_n\| \leqslant \frac{B\, h_0^{2^n}}{\gamma(1 - h_0^{2^n})} \text{ for } n = 0, 1, 2, \ldots;$$

(v) $$\|P(x_n)\| \leqslant \frac{h_0^{2^n}}{\gamma} \text{ for } n = 0, 1, 2, \ldots$$

Proof. First we shall prove by mathematical induction the following relations:

(2) $$x_k \in S(x_0, \delta),$$

(3) $$\|x_k - x_{k-1}\| \leqslant \frac{B}{\gamma} h_0^{2k-1},$$

(4) $$\|P(x_k)\| \leqslant \gamma\|P(x_{k-1})\|^2 \quad \text{for all } k = 1, 2, 3, \ldots.$$

a) From 3° for $n = 0$ results

$$\|x_1 - x_0\| \leqslant B\|P(x_0)\| = \frac{B\eta_0\gamma}{\gamma} \leqslant \frac{B\, h_0}{\gamma(1 - h_0)} < \delta,$$

therefore $x_1 \in S(x_0, \delta)$ ((2) is ture for $k = 1$).

From 3° for $n = o$ also results

$$\|x_1 - x_0\| \leqslant B\|P(x_0)\| = \frac{B\eta_0\gamma}{\gamma} = \frac{B\, h_0}{\gamma},$$

which means that (3) is true for $k = 1$.

We have

$$\|x_0 - u_0\| = \|x_0 - \Phi(x_0)\| = \|P(x_0)\| = \eta_0 \leqslant \frac{\eta_0\gamma}{\gamma(1-h_0)} = \frac{h_0}{\gamma(1-h_0)},$$

therefore $u_0 \in S(x_0, \delta)$.

Using 1°–3° we may write

$$\|P(x_1)\| \leqslant \|P(x_1) - P(u_0) - [x_0, u_0; P](x_1 - u_0)\| + \|P(u_0) +$$
$$+ [x_0, u_0; P](x_1 - u_0)\| \leqslant \|[x_1, x_0, u_0; P](x_1 - x_0)(x_1 - u_0)\| +$$
$$+ \|P(x_0)\|^2 A \leqslant M\|x_1 - x_0\| \cdot \|x_1 - u_0\| + A\|P(x_0)\|^2 \leqslant$$
$$\leqslant (MB^2 + A)\|P(x_0)\|^2 = \gamma\|P(x_0)\|^2,$$

and so fore $k = 1$ (4) is also true.

b) We presume that (2)–(4) are true for any $k \leqslant m$, $m > 1$ and we prove that they are true for $k = m + 1$.

c) From 3° for $n = m$ we have

$$\|x_{m+1} - x_m\| \leqslant B\|P(x_m)\|,$$

which together with (4) lead us to

$$\|x_{m+1} - x_m\| \leqslant \frac{1}{\gamma}\gamma B\|P(x_m)\| \leqslant \frac{1}{\gamma} B\gamma^2\|P(x_{m-1})\|^2 = \frac{B}{\gamma}\|\gamma P(x_{m-1})\|^2 \leqslant$$
$$\leqslant \frac{B}{\gamma}\|\gamma P(x_{m-2})\|^{2^2} \leqslant \ldots \leqslant \frac{B}{\gamma}\|\gamma P(x_0)\|^{2^m} = \frac{B}{\gamma} h_0^{2^m},$$

and so (3) is proved for $k = m + 1$.

By (3) we may writte

$$\|x_{m+1} - x_0\| \leqslant \|x_{m+1} - x_m\| + \ldots + \|x_1 - x_0\| \leqslant$$
$$\leqslant \frac{B}{\gamma}(h_0^{2^m} + h_0^{2^{m-1}} + \ldots + h_0) \leqslant \frac{h_0 B}{\gamma(1-h_0)},$$

which means $x_{m+1} \in S(x_0, \delta)$.

Further we have

$$\|u_m - x_0\| = \|\Phi(x_m) - x_0\| \leqslant \|-x_m + \Phi(x_m)\| + \|x_m - x_{m-1}\| +$$
$$+ \|x_{m-1} - x_{m-2}\| + \ldots + \|x_1 - x_0\| \leqslant \frac{h_0^{2^m}}{\gamma(1-h_0)} + \frac{h_0 B}{\gamma(1-h_0)} \leqslant \frac{h_0(1+B)}{\gamma(1-h_0)},$$

therefore u_m is also in the ball $S(x_0, \delta)$.

Using 1°—3° for $n = m$ we obtain

$$\|P(x_{m+1})\| \leqslant \|P(x_{m+1}) - P(u_m) - [x_m, u_m; P](x_{m+1} - u_m)\| + \\ + \|P(u_m) + [x_m, u_m; P](x_{m+1} - u_m)\| \leqslant \\ \leqslant A\|P(x_m)\|^2 + \|[x_{m+1}, x_m, u_m; P](x_{m+1} - x_m)(x_{m+1} - u_m)\| \leqslant \\ \leqslant A\|P(x_m)\|^2 + MB^2\|P(x_m)\|^2 = (A + MB^2)\|P(x_m)\|^2 = \gamma\|P(x_m)\|^2$$

and so (4) is proved for $k = m + 1$.

We have thus proved that the relations (2)—(4) are true for any positive integer k.

(4) shows that the sequence is second order with respect to the mapping P. Let's prove that the sequence (x_n) is a Cauchy sequence. Using (3) it results:

$$(5)\quad \|x_{n+p} - x_n\| \leqslant \|x_{n+p} - x_{n+p-1}\| + \ldots + \|x_{n+1} - x_n\| \leqslant \\ \leqslant \frac{B}{\gamma}(h_0^{2^{n+p-1}} + \ldots + h_0^{2^n}) \leqslant \frac{B}{\gamma} h_0^{2^n}(1 + h_0^{2^n} + h_0^{2^{n+1}} + \ldots) = \frac{Bh_0^{2^n}}{\gamma(1 - h_0^{2^n})},$$

therefore $\|x_{n+p} - x_n\| \to 0$ when $n \to \infty$ for any p which means that the sequence (x_n) is a Cauchy sequence. The sequence is convergent because X is a Banach space. So the sequence (x_n) is convergent of second order with respect to the mapping P.

From (2) results that $x^* \in S(x_0, \delta)$ and so (ii) is true.

From (5) for $p = 1$ we obtain (iii).

For $p \to \infty$ (5) leads us to the inequality (iv).

From (4) we obtain successively

$$\gamma\|P(x_n)\| \leqslant \gamma^2\|P(x_{n-1})\|^2 = (\gamma\|P(x_{n-1})\|)^2 \leqslant (\gamma^2\|P(x_{n-2})\|^2)^2 = \\ = (\gamma\|P(x_{n-2})\|)^{2^2} \leqslant \ldots \leqslant (\gamma\|P(x_0)\|)^{2^n} = h_0^{2^n},$$

and so (v) is proved.

We only have to prove now that x^* is a solution of equation (1). From (v) for $n \to \infty$ we have

$$\lim_{n\to\infty} \|P(x_n)\| = \|\lim_{n\to\infty} P(x_n)\| = \|P(x^*)\| = 0 \quad (h_0 < 1)$$

therefore $P(x^*) = \theta$.

Thus the theorem has been proved.

Let x_0 be a point of the space X and let's presume that in a neighbourhood of x_0 the mapping $[z_1, z_2; P]^{-1}$ exists. Let's consider the sequence (x_n) given by recurence with the following formula:

$$(6)\quad x_{n+1} = x_n - [x_n, u_n; P]^{-1}P(x_n), \quad u_n = \Phi(x_n), \quad n = 0, 1, 2, \ldots.$$

(Steffensen's method, [1] and [5]).

Consequence. *If the point $x_0 \in X$, the mapping P and the number $r > 0$ are so that in the ball $S(x_0, r) = \{\|x - x_0\| < r\}$ the following conditions are satisfied*

I° $\sup\{\|[z_1, z_2, z_3; P]\| : z_1, z_2, z_3 \in S(x_0, r)\} \leqslant M < +\infty$;

II° *there existsa positive number B_0 so that*

$$\sup\{\|[z_1, z_2; P]^{-1}\| : z_1, z_2 \in S(x_0, r)\} \leqslant B_0;$$

III° $\bar{h}_0 = \|P(x_0)\|M(1 + B_0)^2 = \eta_0\omega < 1$,

$$r = \frac{\bar{h}_0(2 + B_0)}{\omega(1 - \bar{h}_0)},$$

then

j) *the sequence (x_n) is convergent of second order with respect to the mapping P and the limit x^* of the sequence is a solution of equation* (1);

jj) $x^* \in S(x_0, r)$;

(jjj) $\|x_{n+1} - x_n\| \leqslant \dfrac{(1 + B_0)\bar{h}_0^{2^n}}{\omega}$ *for all* $n = 0, 1, 2, \ldots$;

jv) $\|x^* - x_n\| \leqslant \dfrac{(1 + B_0)\bar{h}_0^{2^n}}{\omega(1 - \bar{h}_0^{2^n})}$ *for all* $n = 0, 1, 2, \ldots$;

v) $\|P(x_n)\| \leqslant \dfrac{1}{\omega}\bar{h}_0^{2^n}$ *for all* $n = 0, 1, 2, \ldots$.

Proof. We shall prove that the former theorem can be applied. The conditions 1° and I° are the same. The condition 2° is satisfied for $A = 0$. Really using (6) we have:

$$P(u_n) + [x_n, u_n; P](x_{n+1} - u_n) = P(u_n) + [x_n, u_n; P](x_{n+1} - \\ - x_n + x_n - u_n) = P(u_n) - [x_n, u_n; P][x_n, u_n; P]^{-1}P(x_n) + \\ + [x_n, u_n; P](x_n - u_n) = P(u_n) - P(x_n) + P(x_n) - P(u_n) = \theta.$$

From (6) using II° results

$$\|x_{n+1} - x_n\| \leqslant B_0\|P(x_n)\|.$$

Further we have

$$x_{n+1} - u_n = x_{n+1} - \Phi(x_n) = x_n - [x_n, u_n; P]^{-1}P(x_n) - \Phi(x_n) = \\ = x_n - \Phi(x_n) - [x_n, u_n; P]^{-1}P(x_n) = P(x_n) - [x_n, u_n; P]^{-1}P(x_n),$$

from which we obtain

$$\|x_{n+1} - u_n\| \leqslant (1 + B_0)\|P(x_n)\|.$$

So for $B = 1 + B_0$ the condition 3° is satisfied.

The condition III° is the same with 4° if we take $A = 0$. The notation $\omega = M(1 + B_0)^2$ has been used.

The conditions of the theorem being satisfied we may use it and we obtain j)—v).

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