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ON SOME BIVARIATE SPLINE OPERATORS

by

P. BLAGA and GH. COMAN

(Cluj-Napoca)

Introduction. In some previous papers [9, 6, 5] it was studied the spline operator S_Δ as a generalization of Bernstein's operator which associates to a function f, defined on the interval $[a, b]$, the approximation

$$(1)\qquad (S_\Delta f)(x) = \sum_{i=1}^{m+k} N_i(x) f(\xi_i),\ m > 0,\ k > 0,$$

where $\Delta = \{x_i\}_{-k}^{m+k}$, with

$$(2)\qquad x_{-k} = \ldots = x_0 = a < x_1 \leqslant \ldots \leqslant x_{m-1} < b = x_m = \ldots = x_{m+k}$$

and

$$x_{i-k} < x_i,\quad i = k+1, \ldots, m-1,$$

$$(3)\qquad \xi_i = \frac{x_{i-k} + \ldots + x_{i-1}}{k},\quad i = 1, \ldots, m+k,$$

$$(4)\qquad N_i(x) = \frac{x_i - x_{i-k-1}}{k+1} M_i(x),\quad i = 1, \ldots, m+k$$

and where $M_i(x) = [x_{i-k-1}, \ldots, x_i;\ (k+1)(\cdot - x)_+^k]$ is the $(k+1)$-th divided difference of the function $M(x, t) = (k+1)(t - x)_+^k$ with respect to the variable t.

The approximation $S_\Delta f$ is a spline function of the degree k having the knots at the points of Δ.

Using the operator S_Δ and the method presented in [12] there are constructed some bivariate approximation schemes on a triangle, problem also mentioned in [13]. As an application of these schemes are given some cubature formulas on a triangle.

1. Let G_1, G_2 be two polynomials with the properties $G_2(x) \geqslant G_1(x) \geqslant 0$ for $x \in [0, h]$, $h > 0$, $D = \{(x, y) \in \mathbf{R}^2 | 0 \leqslant x \leqslant h,$ $G_1(x) \leqslant y \leqslant G_2(x)\}$ and $y = \frac{G_2 - G_1}{h} t + G_1$ an application which transform the domain D in the square $[0, h] \times [0, h]$. If one applies to the function $f: D \to \mathbf{R}$ the operator S_Δ with to the first variable, we obtain

$$(S_\Delta f)\left(x, \frac{G_2(x) - G_1(x)}{h} t + G_1(x)\right) = \sum_{i=1}^{m+k} N_i(x) f\left[\xi_i, \frac{G_2(\xi_i) - G_1(\xi_i)}{h} t + G_1(\xi_i)\right].$$

Now, appliyng to the function $f\left[\xi_i, \frac{G_2(\xi_i) - G_1(\xi_i)}{h} t + G_1(\xi_i)\right]$ the operator S_{Δ_i} with

$$\Delta_i : t^i_{-s_i} = \ldots = t^i_0 = 0 < t^i_1 \leqslant \ldots \leqslant t^i_{n_i-1} < h = t^i_{n_i} = \ldots = t^i_{n_i+s_i},$$

$$t^i_j < t^i_{j+s_i}, \; j = 1, \ldots, n_i - 1, \; i = 1, \ldots, m + k,$$

we have

(5)
$$(Sf)(x, y) = \sum_{i=1}^{m+k} \sum_{j=1}^{n_i+s_i} N_i(x) N^i_j\left(\frac{y - G_1}{G_2 - G_1} h\right) f\left[\xi_i, \frac{G_2(\xi_i) - G_1(\xi_i)}{h} \eta^i_j + G_1(\xi_i)\right],$$

where

$$\eta^i_j = \frac{t^i_{-s_i+j} + \ldots + t^i_{j-1}}{s_i}$$

and

$$N^i_j(t) = \frac{t^i_j - t^i_{-s_i+j-1}}{s_i + 1}\left[t^i_{-s_i+j-1}, \ldots, t^i_j;\; (s_i + 1)(\cdot - t)^{s_i}_+\right]$$

for $j = 1, \ldots, n_i + s_i$, $i = 1, \ldots, m + k$.

Lemma 1. *If $G_1, G_2 \in \mathbf{P}_1$, then $Sg = g$, $g \in \mathbf{P}_1$, i.e. $Sg = g$, $g = 1, x, y$.*

Proof. Using the identities

$$\sum_{i=1}^{m+k} N_i(x) = 1, \qquad \sum_{i=1}^{m+k} \xi_i N_i(x) = x,$$

we obtain for $g(x, y) = 1, x, y$ respectively

$$(Sg)(x, y) = \sum_{i=1}^{m+k} N_i(x) \sum_{j=1}^{n_i+s_i} N^i_j\left(\frac{y - G_1}{G_2 - G_1} h\right) = 1,$$

$$(Sg)(x, y) = \sum_{i=1}^{m+k} \xi_i N_i(x) \sum_{j=1}^{n_i+s_i} N^i_j\left(\frac{y - G_1}{G_2 - G_1} h\right) = x$$

and taking into account the liniarity of G_1, G_2,

$$(Sg)(x, y) = \sum_{i=1}^{m+k} N_i(x) \sum_{j=1}^{n_i+s_i} N^i_j\left(\frac{y - G_1(\xi_i)}{G_2(\xi_i) - G_1(\xi_i)} h\right)\left[\frac{G_2(\xi_i) - G_1(\xi_i)}{h} \eta^i_j + G_1(\xi_i)\right] =$$

$$= \sum_{i=1}^{m+k} N_i(x)\left[\frac{G_2(\xi_i) - G_1(\xi_i)}{h} \cdot \frac{y - G_1(\xi_i)}{G_2(\xi_i) - G_1(\xi_i)} h + G_1(\xi_i)\right] = y.$$

The operator S generates the approximation formula

(6)
$$f = Sf + Rf$$

where R is the remainder operator.

Taking in (6) some particular functions G_1, G_2 and some convenient partitions Δ, Δ_i there are obtain approximation formulas for various domains. Such, if $G_1 = 0$, $G_2 = h$ and

$$\Delta_i : y_{-s} = \ldots = y_0 = 0 < y_1 \leqslant \ldots \leqslant y_{n-1} < h = y_n = \ldots = y_{n+s}, \quad i = 1, \ldots, m + k,$$

$$y_j < y_{j+s}, \; j = 1, \ldots, n - 1,$$

then the operator (5) becomes

$$(Sf)(x, y) = \sum_{i=1}^{m+k} \sum_{j=1}^{n+s} N_i(x) \tilde{N}_j(y) f(\xi_i, \eta_j)$$

studied in [2].

If $G_1(x) = 1$, $G_2(x) = h - x$, then the domain D is transformed in the triangle $T_h = \{(x, y) \in \mathbf{R}^2 | x \geqslant 0, y \geqslant 0, x + y \leqslant h\}$ and the operator (5) becomes

(7)
$$(Sf)(x, y) = \sum_{i=1}^{m+k} \sum_{j=1}^{n_i+s_i} N_i(x) N^i_j\left(\frac{hy}{h - x}\right) f\left(\xi_i, \frac{h - \xi_i}{h} \eta^i_j\right).$$

Respecting to the remainder term of the formula (6), in this case, we have

L e m m a 2. *If* $f \in B_{1,1}(T_h)$ [8], *then the remainder term of the formula* (6) *has the representation*

$$(Rf)(x, y) = \int_0^h K_{20}(x, y; s) f^{(2,0)}(s, 0) ds + \int_0^h K_{02}(x, y; t) f^{(0,2)}(0, t) dt +$$

$$+ \iint_{T_h} K_{11}(x, y; s, t) f^{(1,1)}(s, t) ds dt,$$

where

$$K_{20}(x, y; s) = R_{(x,y)}[(x - s)_+]$$
$$K_{02}(x, y; t) = R_{(x,y)}[(y - t)_+]$$
$$K_{11}(x, y; s, t) = R_{(x,y)}[(x - s)_+^0 (y - t)_+^0].$$

Proof. Using Taylor's formula with the integral representation of the remainder term [8], for $p = q = 1$, and taking into account the lemma 1, we obtain

$$(Rf)(x, y) = R\Big[\int_0^h (x - s)_+ f^{(2,0)}(s, 0) ds + \int_0^h (y - t)_+ f^{(0,2)}(0, t) dt +$$

$$+ \iint_{T_h} (x - s)_+^0 (y - t)_+^0 f^{(1,1)}(s, t) ds dt\Big] = \int_0^h R_{(x,y)}[(x - s)_+] f^{(2,0)}(s, 0) ds +$$

$$+ \int_0^h R_{(x,y)}[(y - t)_+] f^{(0,2)}(0, t) dt + \iint_{T_h} R_{(x,y)}[(x - s)_+^0 (y - t)_+^0] f^{(1,1)}(s, t) ds dt.$$

2. In the next it will be studied the operator (7) for some given partitions Δ and Δ_i.

For $m = 1$, $n_i = 1$, $s = k - i + 1$, the operator (7) becomes the Bernstein operator on the triangle T_h [12].

Let Δ and Δ_i be given uniform partitions, i.e.

$$x_{-k} = \ldots = x_0 = 0,\ x_1 = \frac{h}{m},\ \ldots,\ x_{m-1} = \frac{m-1}{m} h,\ x_m = \ldots = x_{m+k} = h$$

and respectively

$$t_{-k}^i = \ldots = t_0^i = 0,\ t_1^i = \frac{h}{m - i + 1},\ \ldots,\ t_{m-i}^i = \frac{m - i}{m - i + 1} h,\ t_{m-i+1}^i = \ldots =$$
$$= t_{m-i+k+1}^i = h, \qquad i = 1, \ldots, m,$$

$$t_{-k+p}^i = \ldots = t_0^i = 0,\ t_1^i = \ldots = t_{k-p+1}^i = h, p = 1, \ldots, k, i = m + 1, \ldots$$
$$\ldots, m + k,$$

It follows that

$$(8) \qquad \xi_i = \begin{cases} \dfrac{i(i-1)}{2mk} h, & i = 1, \ldots, p \\ \dfrac{p(2i - p - 1)}{2mk} h, & i = p + 1, \ldots, \left[\dfrac{m + k + 1}{2}\right] \\ h - \xi_{m+k-i+1}, & i = \left[\dfrac{m + k + 3}{2}\right], \ldots, m + k,\ p = \min\{m, k\}, \end{cases}$$

and respectively

$$(9) \qquad \eta_j^i = \begin{cases} \dfrac{j(j-1)}{2k(m - i + 1)} h, & j = 1, \ldots, p \\ \dfrac{p(2i - p - 1)}{2k(m - i + 1)} h, & j = p + 1, \ldots, \left[\dfrac{m - i + k + 2}{2}\right] \\ h - \eta_{m-i+k-j+2}^i, & j = \left[\dfrac{m - i + k + 4}{2}\right], \ldots, m - i + k + 1, \\ & p = \min\{m - i + 1, k\} \end{cases}$$

for $i = 1, \ldots, m$ and

$$\eta_j^{m+i} = \frac{j - 1}{k - i + 1} h,\ j = 1, \ldots, k - i + 1,\ i = 1, \ldots, k.$$

The explicit form of the operator S will be given in some practical cases.

1°. *The linear-linear case* ($k = 1$, $m \geq 2$). Taking into account (8), (9) there are obtain

$$\xi_i = \frac{i - 1}{m} h,\ \frac{h - \xi_i}{h} \eta_j^i = \frac{j - 1}{m} h, \qquad j = 1, \ldots, m - i + 2;$$
$$i = 1, \ldots, m + 1$$

and the operator (7) becomes

$$(10) \qquad (Sf)(x, y) = \sum_{i=1}^{m+1} \sum_{j=1}^{m-i+2} N_i(x) N_j^i \left(\frac{hy}{h - x}\right) f\left(\frac{i - 1}{m} h, \frac{j - 1}{m} h\right),$$

where

$$N_1(x) = \frac{1}{h}(h - mx)_+$$

$$N_i(x) = \frac{1}{h}\{[(i-2)h - mx]_+ - 2[(i-1)h - mx]_+ + (ih - mx)_+\},$$
$$i = 2, \ldots, m,$$

$$N_{m+1}(x) = \frac{1}{h}[mx - (m-1)h]_+$$

and

$$N_1^i\left(\frac{hy}{h-x}\right) = \frac{1}{h-x}[h - x - (m-i+1)y]_+$$

$$N_j^i\left(\frac{hy}{h-x}\right) = \frac{1}{h-x}\{[(j-2)(h-x) - (m-i+1)y]_+ -$$
$$- 2[(j-1)(h-x) - (m-i+1)y]_+ + [j(h-x) - (m-i+1)y]_+\},$$
$$j = 2, \ldots, m-i+1,$$

$$N_{m-i+2}^i\left(\frac{hy}{h-x}\right) = \frac{1}{h-x}[(m-i+1)y - (m-i)(h-x)]_+, \; i = 1, \ldots, m,$$

$$N_1^{m+1}\left(\frac{hy}{h-x}\right) = 1.$$

The remainder term of the approximation formula generated by the operator (10) can be obtained by lemma 2.

For a detailed study, let us consider $m = 2$. In this case we have

(11)
$$f(x, y) = \frac{(h-2x)_+(h-x-2y)_+}{h(h-x)}f(0, 0) + \frac{2(h-2x)_+[h-x-y-(h-x-2y)_+]}{h(h-x)}f\left(0, \frac{h}{2}\right) +$$
$$+ \frac{(h-2x)_+(x+2y-h)_+}{h(h-x)}f(0, h) + \frac{2[h-x-(h-2x)_+](h-x-y)}{h(h-x)}f\left(\frac{h}{2}, 0\right) +$$
$$+ \frac{2[h-x-(h-2x)_+]y}{h(h-x)}f\left(\frac{h}{2}, \frac{h}{2}\right) + \frac{(2x-h)_+}{h}f(h, 0) + (Rf)(x, y),$$

where by lemma 2

$$(Rf)(x, y) = \int_0^h K_{20}(x, y; s)f^{(2,0)}(s, 0)ds + \int_0^h K_{02}(x, y; t)f^{(0,2)}(0, t)dt +$$

(12)
$$\iint_{T_h} K_{11}(x, y; s, t)f^{(1,1)}(s, t)dsdt,$$

with

$$K_{20}(x, y; s) = (x - s)_+ - \frac{1}{h}[h - x - (h-2x)_+](h-2s)_+ -$$

(13)
$$- \frac{1}{h}(2x-h)_+(h-s), \; K_{02}(x, y; t) = (y-t)_+ -$$
$$- \frac{(h-x)y - (h-2x)_+(x+2y-h)_+}{h(h-x)}(h-2t)_+ - \frac{(h-2x)_+(x+2y-h)_+}{h(h-x)}(h-t),$$

$$K_{11}(x, y; s, t) = (x-s)_+^0(y-t)_+^0 - \frac{2[h-x-(h-2x)_+]y}{h(h-x)}(h-2s)_+^0(h-2t)_+^0$$

THEOREM 1. *If* $f \in B_{1,1}^{2,\infty}(T_h)$, *then*

$$\|Rf\| < \frac{h^2}{32}\|f^{(2,0)}(\cdot, 0)\| + \frac{h^2}{16}\|f^{(0,2)}(0, \cdot)\| + \frac{h^2}{2}\|f^{(1,1)}\|,$$

where $\|\cdot\|$ *is the uniform norm.*

Proof. In order to apply the medium theorem one studies the functions K_{20}, K_{02} and K_{11} sign.

Let us consider first the function K_{20};

A. $s \leqslant \frac{h}{2}$

$$K_{20}(x, y; s) = \begin{cases} -\frac{x(h-2s)}{h} & \text{for } x \leqslant s \\ -\frac{(h-2x)_+ s}{h} & \text{for } x > s. \end{cases}$$

It follows that $K_{20}(x, y; s) \leqslant 0$, $(x, y) \in T_h$.

B. $s > \frac{h}{2}$

$$K_{20}(x, y; s) = \begin{cases} -\frac{1}{h}(2x-h)_+(h-s) & \text{for } x \leqslant s \\ \frac{(h-x)(h-2s)}{h} & \text{for } x > s, \end{cases}$$

such that $K_{20}(x, y; s) \leqslant 0$, $(x, y) \in T_h$.

In an analogous way one obtains that $K_{02}(x, y; t) \leqslant 0$, $(x, y) \in T_h$, $t \in [0, h]$.

It is easy seen that the function K_{11} changes the sign on the domain T_h.

In this way we have

$$(Rf)(x, y) = f^{(2,0)}(\xi, 0)\varphi_{20}(x, y) + f^{(0,2)}(0, \eta)\varphi_{02}(x, y) + \iint_{T_h} K_{11}(x, y; s, t) f^{(1,1)}(s, t) ds dt,$$

where $\xi, \eta \in [0, h]$ and

$$\varphi_{20}(x, y) = \begin{cases} \frac{x^2}{2} - \frac{h(3x-h)}{4} & \text{for } x \geq \frac{h}{2} \\ \frac{x^2}{2} - \frac{hx}{4} & \text{for } x < \frac{h}{2} \end{cases},$$

$$\varphi_{02}(x, y) = \begin{cases} \frac{y(2y-h)}{4} - \frac{h(h-2x)(x+2y-h)}{4(h-x)}, & x \leq \frac{h}{2} \text{ and } x + 2y \geq h \\ \frac{y(2y-h)}{4}, & \text{otherwise}, \end{cases}$$

It follows that

(14) $$\max_{T_h} |\varphi_{20}(x, y)| = \frac{h^2}{32}, \quad \max_{T_h} |\varphi_{02}(x, y)| = \frac{73 - 17\sqrt{17}}{64} h^2 < \frac{1}{16} h^2.$$

Hence

$$|(Rf)(x, y)| < \frac{h^2}{32} \|f^{(2,0)}(\cdot, 0)\| + \frac{h^2}{16} \|f^{(0,2)}(0, \cdot)\| + \|f^{(1,1)}\| \iint_{T_h} |K_{11}(x, y; s, t)| ds dt.$$

Taking into account that $|K_{11}(x, y; s, t)| \leq 1$ on T_h the proof follows.

2°. *The Bernstein-linear case.* Using the partitions

$$\Delta : x_{-m} = \ldots = x_0 = 0 < h = x_1 = \ldots = x_{m+1},$$

$$\Delta_i : t^i_{-1} = t^i_0 = 0, \; t^i_j = \frac{jh}{m - i + 1}, \; j = 1, \ldots, m - i, \; t^i_{m-i+1} = t^i_{m-i+2} = h,$$

$$i = 1, \ldots, m$$

$$\Delta_{m+1} : t^{m+1}_0 = 0, \quad t^{m+1}_1 = h,$$

one obtains

$$\xi_i = \frac{i-1}{m} h, \frac{h - \xi_i}{h} \eta^i_j = \frac{j-1}{m} h, \; i = 1, \ldots, m+1, \; j = 1, \ldots, m - i + 2$$

and

$$N_i(x) = \binom{m}{i-1} \left(\frac{x}{h}\right)^{i-1} \left(\frac{h-x}{h}\right)^{m-i+1}, \quad i = 1, \ldots, m+1,$$

N^i_j having the expressions from the previous case.

We remark that S, in this case, is a spline polynomial operator.

In detail will be studied the practical case $m = 2$. One obtaines the approximation formula

$$f(x, y) = \frac{1}{h^2}(h - x)(h - x - 2y)_+ f(0, 0) + \frac{2}{h^2}(h - x)[h - x - y -$$

(15) $$- (h - x - 2y)_+] f\left(0, \frac{h}{2}\right) + \frac{1}{h^2}(h - x)(x + 2y - h)_+ f(0, h) +$$

$$+ \frac{2}{h^2} x(h - x - y) f\left(\frac{h}{2}, 0\right) + \frac{2}{h^2} xy f\left(\frac{h}{2}, \frac{h}{2}\right) + \frac{x^2}{h^2} f(h, 0) + (Rf)(x, y),$$

where

$$(Rf)(x, y) = \int_0^h K_{20}(x, y; s) f^{(2,0)}(s, 0) ds + \int_0^h K_{02}(x, y; t) f^{(0,2)}(0, t) dt + \iint_{T_h} K_{11}(x, y; s, t) f^{(1,1)}(s, t) ds dt,$$

with

$$K_{20}(x, y; s) = (x - s)_+ - \frac{x(h-x)}{h^2}\left[(h - 2s)_+ - \frac{x^2}{h^2}(h - s)\right],$$

$$K_{02}(x, y; t) = (y - t)_+ - \frac{1}{h^2}\{(h - x)[h - x - y - (h - x - 2y)_+] + xy\}(h - 2t)_+ - \frac{h - x}{h^2}(x + 2y - h)_+(h - t),$$

$$K_{11}(x, y; s, t) = (x - s)^0_+ (y - t)^0_+ - \frac{2xy}{h^2}\left(\frac{h}{2} - s\right)^0_+ \left(\frac{h}{2} - t\right)^0_+.$$

THEOREM 2. *If* $f \in B^{2,\infty}_{1,1}(T_h)$, *then*

$$\|Rf\| \leq \frac{h^2}{16} \|f^{(2,0)}(\cdot, 0)\| + \frac{h^2}{16} \|f^{(0,2)}(0, \cdot)\| + \frac{h^2}{2} \|f^{(1,1)}\|.$$

Proof. As in the previous case the functions K_{20} and K_{02} are negative and the function K_{11} changes the sign on T_h. So we have

$$(Rf)(x, y) = \varphi_{20}(x, y) f^{(2,0)}(\xi, 0) + \varphi_{02}(x, y) f^{(0,2)}(0, \eta) + \iint_{T_h} K_{11}(x, y; s, t) f^{(1,1)}(s, t) ds dt,$$

with $\xi, \eta \in [0, h]$, where

$$\varphi_{20}(x, y) = -\frac{x(h-x)}{4}, \quad \varphi_{02}(x, y) = \frac{y(2y-h)}{4} - \frac{(h-x)(x+2y-h)_+}{4}.$$

It follows that

$$\max_{T_h} |\varphi_{20}(x, y)| = \frac{h^2}{16}, \quad \max_{T_h} |\varphi_{02}(x, y)| = \frac{h^2}{16}, \quad \max_{T_h} |K_{11}(x, y; x, t)| = 1$$

and the theorem is proved.

3. Next, using the approximation formulas (11) and (15) there are constructed some cubature formulas on the triangle T_h.

If we integrate on T_h each member of the formula (10), one obtains

$$\iint_{T_h} f(x, y) dx dy = \frac{h^2}{96}\left[5f(0,0) + 10f\left(0, \frac{h}{2}\right) + 5f(0, h) + 12f\left(\frac{h}{2}, 0\right) + 12f\left(\frac{h}{2}, \frac{h}{2}\right) + 4f(h, 0)\right] + R(f),$$

where

$$R(f) = C_{20} f^{(2,0)}(\xi, 0) + C_{02} f^{(2,0)}(0, \eta) + \iint_{T_h} \varphi_{11}(s, t) f^{(1,1)}(s, t) ds dt,$$

with

$$C_{20} = \iint_{T_h} \varphi_{20}(x, y) dx dy = -\frac{1}{96} h^4,$$

$$C_{02} = \iint_{T_h} \varphi_{02}(x, y) dx dy = -\frac{5}{384} h^4,$$

$$\varphi_{11}(s, t) = \iint_{T_h} K_{11}(x, y; s, t) dx dy = \frac{(h-s-t)^2}{2} - \frac{h^2}{8}(h-2s)_+^0 (h-2t)_+^0.$$

Taking into account that

$$\iint_{T_h} |\varphi_{11}(s, t)| ds dt = \frac{13}{384} h^4$$

we obtain

$$|R(f)| \leqslant \frac{h^4}{384}[4\|f^{(2,0)}(\cdot, 0)\| + 5\|f^{(0,2)}(0, \cdot)\| + 13\|f^{(1,1)}\|].$$

In an analogous way, using the formula (15), we obtain

$$\iint_{T_h} f(x, y) dx dy = \frac{h^2}{48}\left[3f(0, 0) + 6f\left(0, \frac{h}{2}\right) + 3f(0, h) + 4f\left(\frac{h}{2}, 0\right) + 4f\left(\frac{h}{2}, \frac{h}{2}\right) + 4f(h, 0)\right] + R(f),$$

where

$$|R(f)| \leqslant \frac{h^4}{384}[8\|f^{(2,0)}(\cdot, 0)\| + 6\|f^{(0,2)}(0, \cdot)\| + \|f^{(1,1)}\|].$$

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