MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 8, N° 2, 1979, pp. 177—180

THE PRINCIPLE OF THE MAJORANT IN SOLVING OPERATOR EQUATIONS WHICH DEPEND ON PARAMETERS

by

SEVER GROZE
(Cluj-Napoca)

1. Let us consider the operator equation

$$P(x, \alpha) = 0, \tag{1}$$

where $P: X \times M \to X$, M is a normed linear space, and X is a Fréchet space [1], P being continuous.

The existence and unicity of the solution $x^*(\alpha)$ of the equation (1), using the iterative method

$$x_{n+1} = x_n - \Gamma^n_\alpha P(x_n, \alpha), \tag{2}$$

where $\Gamma^n_\alpha = [P'(x_n, \alpha)]^{-1}$, was studied, using the principle of the majorant of I. V. KANTOROVICH [2], in the case of a complete normed linear space X. This study was taken again by B. JANKÓ [3], using the method of successive approximations and the generalized method of Newton-Kantorovich, where X is a space as above.

In both cases the Fréchet derivative of some order is used, to solv the problem.

In this paper we take again this study, considering a complete supermetric space X, using the concept of divided difference, which is much more general than the notion of Fréchet derivative,

$$x_{n+1} = x_n - [P^{(\alpha)}_{x_n, x_{n-1}}]^{-1} P(x_n, \alpha) \tag{2'}$$

2. Let $P: X \times M \to X$ be a nonlinear operator which has partial divided differences in respect to x and α, X and M being as above.

We denote by $\rho_X(P)$ and $\rho_{X\times M, X}(P^{(\alpha)}_{x^{(1)}, x^{(2)}})$ the generalized norms of the operator P and of its partial divided difference with respect to x, supposing that α constant. We also denote by $\rho_{X\times M,\,X}(P^{(x)}_{\alpha^{(1)},\,\alpha^{(2)}})$ the generalized norm of the partial divided difference of the operator with respect to α, supposing x constant.

For the partial divided difference operator, of a partial divided difference, we will use the notation

$$(P^{(\alpha)}_{x^{(1)},\,x^{(2)}})^{(\alpha)}_{x^{(2)},\,x^{(3)}} = P^{(\alpha^2)}_{x^{(1)},\,x^{(2)},\,x^{(3)}}$$

$$(P^{(\alpha)}_{x^{(1)},\,x^{(2)}})^{(x)}_{\alpha^{(1)},\,\alpha^{(2)}} = P^{(\alpha,\,x)}_{x^{(1)},\,x^{(2)}|\alpha^{(1)},\,\alpha^{(2)}}$$

and for their generalized norms

$$\rho_{(X\times M)^2,\,X}(P^{(\alpha^2)}_{x^{(1)},\,x^{(2)},\,x^{(3)}})$$

$$\rho_{(X\times M),\,X}(P^{(\alpha,\,x)}_{x^{(1)},\,x^{(2)}|\alpha^{(1)},\,\alpha^{(2)}})$$

respectively.

Since they are important for our study we recall here two theorems [4] about the existence and the unicity of the solution of the operator equation

$$x = U(x) \tag{3}$$

obtained by the iteration

$$x_{n+1} = U(x_n). \tag{3'}$$

This equation is majorized, in the sense of the definition given in [2], by the real equation

$$z = V(z) \tag{4}$$

whose solutions are obtained by the iterative method

$$z_{n+1} = V(z) \tag{4'}$$

THEOREM A. *If the equation* (3) *is majorized by the equation* (4) *whose smallest root is* $z^* \in [z, z']$, *then it has a solution* x^*, *verifying the condition*

$$\rho_X(x^* - x_0) \leqslant z^* - z_0 \leqslant z' - z_0$$

and which is the limit of the successive approximations [3']

THEOREM B. *If the continuous operator* $U : X \to X$ *has divided differences,* X *being a complete supermetric space, and* V *is a continuous function defined in* $[z_0, z']$, *and if the conditions*

(i) $\rho_{X,\,X}(U_{x^{(1)},\,x^{(2)}}) \leqslant V_{z^{(1)},\,z^{(2)}},\ \forall x^{(i)}$ for which

$$\rho_X(x^{(i)} - x_0) \leqslant z^{(i)} - z_0 \leqslant z' - z_0$$

(ii) $$V(z_0) \geqslant z_0,\ V(z') \leqslant z'$$

are satisfied, then from the unicity of the root of the equation (4) in (z, z'), it follows the unicity of the solution of the equation (3) from S defined by

$$\rho_X(x - x_0) \leqslant z' - z_0.$$

If this solution exists, it is the limit of the succesive approximation (3'), for any initial approximation $x_0 \in S$.

3. In conection with the operator equation (1) we prove the following

THEOREM. *If the equation* (1) *is majorized by the equation*

$$z = V(z, \beta) \tag{1'}$$

where $V(z, \beta)$ *is a continuous real function defined in the rectangle* $z_0 \leqslant z \leqslant z'$, $\beta_0 \leqslant \beta \leqslant \beta'$, *the conditions*

1°. $\rho_X(x_0 - P(x_0, \alpha_0)) \leqslant V(z_0, \beta_0)) - z_0$

2°. $\rho_{X\times M,\,X}(P^{(\alpha_0)}_{x^{(1)},\,x^{(2)}}) \leqslant V^{(\beta_0)}_{z^{(1)},\,z^{(2)}}$, for

$$\rho_X(x^{(i)} - x_0) \leqslant z^{(i)} - z_0 \leqslant z' - z_0,\ (i = 1, 2)$$

3°. $\rho_{X\times M,\,X}(P^{(x_0)}_{\alpha^{(1)},\,\alpha^{(2)}}) \leqslant V^{(z_0)}_{\beta^{(1)},\,\beta^{(2)}})$

4°. $\rho_{(X\times M)^2,\,X}(P^{(\alpha,\,x)}_{x^{(1)},\,x^{(2)}|\alpha^{(1)},\,\alpha^{(2)}}) \leqslant V^{(\beta,\,z)}_{z^{(1)},\,z^{(2)}|\beta^{(1)},\,\beta^{(2)}}$,

for $$\rho_X(x - x_0) \leqslant z - z_0 \leqslant z' - z_0$$

and $$\rho_M(\alpha - \alpha_0) \leqslant \beta - \beta_0 \leqslant \beta' - \beta_0$$

are satisfied, and assume that equation (1') *has a solution for* $\beta \in [\beta_0, \beta']$, *then the equation* (1) *has a solution* $x^*(\alpha)$ *for* $\alpha \in S$, *defined by* $\rho_M(\alpha - \alpha_0) \leqslant \beta - \beta_0$. *This solution can be computed by the method of succesive approximation. It is unique in* $\rho_X(x - x_0) \leqslant z' - z_0$ *if the equation* (1') *has unique solution* $z^* \in (z_0, z')$.

Proof. To prove the existence of the solution of the equation (1) we use the theoreme A. We shall show that the conditions 1° and 2° hold for any α and β.

$$\rho_X(x_0 - P(x_0, \alpha)) \leqslant \rho_X(x_0 - P(x_0, \alpha_0)) + \rho_X(P(x_0, \alpha) - P(x_0, \alpha_0)). \tag{5}$$

Since

$$P(x_0, \alpha) - P(x_0, \alpha_0) = P^{(x_0)}_{\alpha,\,\alpha_0}(\alpha - \alpha_0),$$

and taking in account the conditions 1°, 3° the relation (5) becomes

$$\rho_X(x_0 - P(x_0,\ \alpha)) \leqslant V(z_0,\ \beta_0) - z_0 + V^{(z_0)}_{\beta,\ \beta_0}(\beta - \beta_0) =$$
$$= V(z_0,\ \beta_0) - z_0 + V(z_0,\ \beta) - V(z_0,\ \beta_0) = V(z_0,\ \beta) - z_0.$$

Then

$$P^{(\alpha)}_{x^{(1)},\ x^{(2)}} = P^{(\alpha_0)}_{x^{(1)},\ x^{(2)}} + P^{(\alpha)}_{x^{(1)},\ x^{(2)}} - P^{(\alpha_0)}_{x^{(1)},\ x^{(2)}} =$$
$$= P^{(\alpha_0)}_{x^{(1)},\ x^{(2)}} + P^{(\alpha,\ x)}_{x^{(1)},\ x^{(2)}|\alpha,\ \alpha_0}(\alpha - \alpha_0)$$

and

$$\rho_{X \times M,\ X}\left(P^{(\alpha)}_{x^{(1)},\ x^{(2)}}\right) \leqslant V^{(\beta_0)}_{z^{(1)},\ z^{(2)}} + V^{(\beta,\ z)}_{z^{(1)},\ z^{(2)}|\beta,\ \beta_0}(\beta - \beta_0) =$$
$$= V^{(\beta_0)}_{z^{(1)},\ z^{(2)}} + V^{(\beta)}_{z^{(1)},\ z^{(2)}} - V^{(\beta)}_{z^{(1)},\ z^{(2)}} = V^{(\beta)}_{z^{(1)},\ z^{(2)}}.$$

Using theorem B all assertions of the present theorem follow imediately.

REFERENCES

[1] Yosida, K., *Funktionalanalysis*. Springer, Berlin — Heidelberg — New York, 1969

[2] Kantorovich, L. V., *Nekatorye dalneisye primenenija metoda N'iutona dlja functional'nyh uravnenija*. Vestnic Leningrad. Univ., 7, 2, 1957.

[3] Jankó, B., *Rezolvarea ecuațiilor operaționale neliniare în spații Banach*. Ed. Acad. R.S.R., 1969.

[4] Groze, S., *Principiul majorantei și rezolvarea ecuațiilor operaționale neliniare definit în spații supermetrice prin metoda aproximațiilor succesive*. Anal. șt. ale Univ „Al. I. Cuza" Iași, Secțiunea I, Tom XVIII, 1(1972).

Received 15. II. 1979.