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THE METHOD OF CHORDS FOR SOLVING OPERATOR EQUATIONS DEPENDENT ON ONE PARAMETER

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Let us consider the operator equation

(1) $$P(x, \alpha) = \theta,$$

where $P: X \times M \to X$, X is a Fréchet space, and M is a quasinormed linear space. Suppose that the equation is majorized by

(2) $$Q(z, \beta) = 0,$$

where Q is a real function of real variables, continous, defined on the rectangle $z \in [z_0, z']$, $\beta \in [\beta_0, \beta']$.

In order to obtain the solution $x(\alpha)$ of the equation (1), we use the iterative method

(1′) $$x_{n+1} = x_n - \Lambda_{n,\alpha} P(x, \alpha) \quad (n = 0, 1, \ldots),$$

where $\Lambda_{n,\alpha} = [P^{\alpha}_{x_n, x_{n-1}}]^{-1}$ is the inverse of the partial divized difference of the operator P with respect to x [1], method known as the method of chords.

To find the solution $z(\beta)$ of the majorant equation, we consider the iteration

(2′) $$z_{n+1} = z_n - \frac{z_n - z_{n-1}}{Q(z_n, \beta) - Q(z_{n-1}, \beta)} Q(z_n, \beta)$$

$$(n = 0, 1, \ldots).$$

In [2] the following theorem was proved

THEOREM A. *If the operator equation* $P(x) = 0$ *has as majorant, the real equation* $Q(z) = 0$, *and if the following properties are valied for the initial approximations* x_{-1}, x_0 *respectively* z_{-1}, z_0, *with* $z_{-1} < z_0$:

1. *There is* $\Lambda_0 = -[P_{x_0, x_{-1}}]^{-1}$ *and* $\rho_{X, X}(\Lambda_0) \leqslant B_0$,

$$B_0 = -\frac{1}{Q_{z_0, z_{-1}}} > 0;$$

2. $\rho_X(P(x_0)) \leqslant Q(z_0)$ *and* $\rho_X(P(x_{-1})) \leqslant Q(z_{-1})$;

3. $\rho_{X, X}(P_{x^{(1)}, x^{(2)}} - P_{x^{(2)}, x^{(3)}}) \leqslant Q_{z^{(1)}, z^{(2)}} - Q_{z^{(2)}, z^{(3)}}$

for any $x^{(i)} \in S$, *where* S *is defined by*

$$\rho_X(x - x_0) \leqslant z - z_0 \leqslant z' - z_0, \; i = 1, 2, 3;$$

then from the existence of the solution $z^* \in [z_0, z']$ *of the real equation* (2), *implies the existence of at least one solution* $x^* \in S$ *of the operator equation, solution which is the limit of the sequence* (x_n) *given by the iterative procedure*

$$x_{n+1} = x_n - \Lambda_n P(x_n), \quad n = 0, 1, \ldots,$$

where $\Lambda_n = [P_{x_n, x_{n-1}}]^{-1}$. *The order of convergence is characterized by*

$$\rho_X(x^* - x_n) \leqslant z^* - z_n.$$

This theorem will be used on proving the existence of the solution of the equation (1). Next we prove

THEOREM 1. *Assume that the following conditions hold*

1. *For the initial approximations* x_{-1}, x_0, *and* z_{-1}, z_0 *respectively, and for the initial values* α_0, β_0 *for the parameters* α, β, *there exists the operator*

$$\Lambda_{0, \alpha_0} = [P^{(\alpha_0)}_{x_0, x_{-1}}]^{-1}, \; \rho_{X \times M, X}(\Lambda_0, \alpha_0) \leqslant -\frac{1}{Q^{(\beta_0)}_{z_{-1}, z_0}} = B_0;$$

2. $\rho_X(P(x_i, \alpha_0)) \leqslant Q(z_i, \beta_0)$, $i = -1, 0$;

3. $\rho_{X \times M, X}(P^{(\alpha_0)}_{x^{(1)}, x^{(2)}} - P^{(\alpha_0)}_{x^{(2)}, x^{(3)}}) \leqslant Q^{(\beta_0)}_{z^{(1)}, z^{(2)}} - Q^{(\beta_0)}_{z^{(2)}, z^{(3)}}$.

$$\forall x^{(i)} \in S, \; S \textit{ being defined by}$$

$$\rho_X(x^{(i)} - x_0) \leqslant z^{(i)} - z_0 \leqslant z' - z_0, \; (i = 1, 2);$$

4. $\rho_X(P^{(x_0)}_{\alpha^{(1)}, \alpha^{(2)}}) \leqslant Q^{(z_0)}_{\beta^{(1)}, \beta^{(2)}}$, $\forall \alpha^{(i)} \in \sigma$, σ *being defined by*

$$\rho_M(\alpha^{(i)} - \alpha_0) \leqslant \beta^{(i)} - \beta_0 \leqslant \beta' - \beta_0, \quad (i = 1, 2,);$$

5. $\rho_{(X \times M), X}(P^{(\alpha, x_0)}_{x^{(1)}, x^{(2)} | \alpha^{(1)}, \alpha^{(2)}}) \leqslant Q^{(\beta, z_0)}_{z^{(1)}, z^{(2)} | \beta^{(1)}, \beta^{(2)}}$

$$\forall x^{(i)} \in S \quad \textit{and} \quad \alpha^{(i)} \in \sigma, \; (i = 1, 2);$$

6. $\rho_{(X \times M)^2, X}(P^{(\alpha, x)}_{x^{(1)}, x^{(2)} | \alpha^{(1)}, \alpha^{(2)}} - P^{(\alpha, x_0)}_{x^{(1)}, x^{(2)} | \alpha^{(1)}, \alpha^{(2)}}) \leqslant Q^{(\beta, z)}_{z^{(1)}, z^{(2)} | \beta^{(1)}, \beta^{(2)}} -$
$- Q^{(\beta, z_0)}_{z^{(1)}, z^{(2)} | \beta^{(1)}, \beta^{(2)}}$

$$\forall x^{(i)} \in S \quad (i = 1, 2), \; \beta \in \sigma.$$

Then from the existence of the root $z^*(\beta) \in [z_0, z']$, *for any* $\beta \in [\beta_0, \beta']$ *of the equation* (2) *to which the procedure* (2') *converges, it résults the existence of a solution* $x^*(\alpha)$, *for any* $\alpha \in \sigma$ *of the equation* (1), *solution to which the iterative procedure* (1') *converges. The order of the convergence is characterized by*

$$\rho_X(x^*(\alpha) - x_0) \leqslant z^*(\beta) - z_0. \tag{3}$$

Proof. One observes that the conditions 1—3 of the theorem 1 can be applied in the case of equations independent of parameters, i.e. for equations such as $P(x, \alpha_0) = 0$ and $Q(z, \beta_0) = 0$, α_0 and β_0 being fixed. For these equations the existence of a solution $x^*(\alpha_0)$ follows by the Theorem A.

We prove that these conditions are satisfied for any $\alpha \in \sigma$ and $\beta \in [\beta_0, \beta']$.

a) Let us consider the operator

$$I + \Lambda_{0, \alpha_0} P^{(\alpha)}_{x_0, x_{-1}} = \Lambda_{0, \alpha_0}(P^{(\alpha)}_{x_0, x_{-1}} - P^{(\alpha_0)}_{x_0, x_{-1}}) =$$

$$= \Lambda_{0, \alpha_0} P^{(\alpha, x)}_{x_0, x_{-1} | \alpha, \alpha_0} (\alpha - \alpha_0).$$

Taking in account the condition 5. , we can write

$$\rho_{(X \times M)^2, X}(I + \Lambda_{0, \alpha_0} P^{(\alpha)}_{x_0, x_{-1}}) \leqslant B_{0, \beta_0} Q^{(\beta, z)}_{z_0, z_{-1} | \beta, \beta_0}(\beta - \beta_0) =$$

$$= B_{0, \beta_0}(Q^{(\beta)}_{z_0, z_{-1}} - Q^{(\beta_0)}_{z_0, z_{-1}}) = 1 - \frac{Q^{(\beta)}_{z_0, z_{-1}}}{Q^{(\beta_0)}_{z_0, z_{-1}}} = q.$$

From the existence of the solution $z^*(\beta) \in [z_0, z']$, $\forall \beta \in [\beta_0, \beta']$ it results $\frac{Q^{(\beta)}_{z_0, z_{-1}}}{Q^{(\beta_0)}_{z_0, z_{-1}}} > 0$, consequently $q < 1$ and from the Banach theorem, it follows the existence of the operator

$$H^{-1} = [I - (I + \Lambda_{0, \alpha_0} P^{(\alpha)}_{x_0, x_{-1}}]^{-1} = [-\Lambda_{0, \alpha_0} P^{(\alpha)}_{x_0, x_{-1}}]^{-1}.$$

Then, it also results the existence of

$$H^{-1}\Lambda_{0,\alpha_0} = [-\Lambda_{0,\alpha_0} P^{(\alpha)}_{x_0, x_{-1}}]^{-1}\Lambda_{0,\alpha_0} = -[P^{(\alpha)}_{x_0, x_{-1}}]^{-1} = \Lambda_{0,\alpha},$$

for which we have

$$\rho_{X\times M, X}(\Lambda_{0,\alpha}) = \rho_{X\times M, X}(H^{-1}\ \Lambda_{0,\alpha}) = \frac{1}{1-q} B_{0,\beta_0} = -\frac{1}{Q^{(\beta)}_{z_0, z_{-1}}} = B_{0,\beta}.$$

b) To prove that condition 2. holds, we consider the equality

$$\rho_X(P(x_0, \alpha)) = \rho_X(P(x_0, \alpha) + P(x_0, \alpha_0) - P(x_0, \alpha_0))$$

which, using the condition 2, may be written

$$\rho_X(P(x_0, \alpha) \leqslant Q(z_0, \beta_0) + \rho_X(P(x_0, \alpha) - P(x_0, \alpha_0)) =$$
$$= Q(z_0, \beta_0) + \rho_X(P^{(x_0)}_{\alpha, \alpha_0}(\alpha - \alpha_0))$$

and due to condition 4, we have

$$\rho_X(P(x_0, \alpha)) \leqslant Q(z_0, \beta_0) + Q^{(z_0)}_{\beta, \beta_0}(\beta - \beta_0) =$$
$$= Q(z_0, \beta_0) + Q(z_0, \beta) - Q(z_0, \beta_0) = Q(z_0, \beta).$$

In the same way one proves

$$\rho_X(P(x_{-1}, \alpha) \leqslant Q(z_{-1}, \beta)$$

c) Let us now consider the operator

$$P^{(\alpha)}_{x^{(1)}, x^{(2)}} - P^{(\alpha)}_{x^{(2)}, x^{(3)}} = P^{(\alpha)}_{x^{(1)}, x^{(2)}} - P^{(\alpha_0)}_{x^{(1)}, x^{(2)}} + P^{(\alpha_0)}_{x^{(1)}, x^{(2)}} - (P^{(\alpha)}_{x^{(2)}, x^{(3)}} -$$
$$- P^{(\alpha_0)}_{x^{(2)}, x^{(3)}} + P^{(\alpha_0)}_{x^{(2)}, x^{(3)}}) = P^{(\alpha, x)}_{x^{(1)}, x^{(2)}|\alpha, \alpha_0} - P^{(\alpha, x)}_{x^{(2)}, x^{(3)}|\alpha, \alpha_0}(\alpha - \alpha_0) +$$
$$+ P^{(\alpha_0)}_{x^{(1)}, x^{(2)}} - P^{(\alpha_0)}_{x^{(2)}, x^{(3)}} = [P^{(\alpha, x)}_{x^{(1)}, x^{(2)}|\alpha, \alpha_0} - P^{(\alpha, x)}_{x^{(2)}, x^{(3)}|\alpha, \alpha_0}](\alpha - \alpha_0) +$$
$$+ P^{(\alpha_0)}_{x^{(1)}, x^{(2)}} - P^{(\alpha_0)}_{x^{(2)}, x^{(3)}}.$$

For the generalized norm we have

$$\rho_{X\times M, X}(P^{(\alpha)}_{x^{(1)}, x^{(2)}} - P^{(\alpha)}_{x^{(2)}, x^{(3)}}) \leqslant \rho_{(X\times M)^2, X}(P^{(\alpha, x)}_{x^{(1)}, x^{(2)}|\alpha, \alpha_0} -$$
$$- P^{(\alpha, x)}_{x^{(2)}, x^{(3)}|\alpha, \alpha_0})\rho_M(\alpha - \alpha_0) + \rho_{(X\times M), X}(P^{(\alpha_0)}_{x^{(1)}, x^{(2)}} - P^{(\alpha_0)}_{x^{(2)}, x^{(3)}})$$

which, due to conditions 6° and 3° gives

$$\rho_{X\times M, X}(P^{(\alpha)}_{x^{(1)}, x^{(2)}} - P^{(\alpha)}_{x^{(2)}, x^{(3)}}) \leqslant (Q^{(\beta, z)}_{z^{(1)}, z^{(2)}|\beta, \beta_0} - Q^{(\beta, z)}_{z^{(2)}, z^{(3)}|\beta, \beta_0}(\beta - \beta_0) +$$
$$+ Q^{(\beta_0)}_{z^{(1)}, z^{(2)}} - Q^{(\beta_0)}_{z^{(2)}, z^{(3)}} = Q^{(\beta)}_{z^{(1)}, z^{(2)}} - Q^{(\beta_0)}_{z^{(1)}, z^{(2)}} - (Q^{(\beta)}_{z^{(2)}, z^{(3)}} - Q^{(\beta_0)}_{z^{(2)}, z^{(3)}}) +$$
$$+ Q^{(\beta_0)}_{z^{(1)}, z^{(2)}} - Q^{(\beta_0)}_{z^{(2)}, z^{(3)}} = Q^{(\beta)}_{z^{(1)}, z^{(2)}} - Q^{(\beta)}_{z^{(2)}, z^{(3)}},$$

which proves that the condition 3° holds.

The assertions of theorem 1 follows from theorem A.

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