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SOME PROPERTIES OF SOLUTIONS OF EQUATION $\Delta^4 u = 0$

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1. In the paper [2] it was studied some properties of triharmonic functions.

The purpose of this note is to do analogous properties for the solutions of equation $\Delta^4 u = 0$, where Δ is Laplace's operator.

For easily writing, and for simplifying the calculations we consider the case of functions of 3 variables, although the results may be, as well as, made for the functions of $n > 3$ variables. Also, instead of the partial derivatives we use the indices $\left(\text{E.q. } \frac{\partial u}{\partial x} = u_x, \text{ etc.}\right)$.

Let $D \subset R^3$ be a bounded domain, with the boundary Γ. We consider the solutions of the equation

$$\Delta^4 u = 0 \tag{1}$$

in the domain D, where Δ is Laplace's operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{2}$$

2. In this section we deduce from the solutions of equation (1) some triharmonic, biharmonic and harmonic functions in the domain D.

It holds the following

THEOREM 1. *If $w = w(x, y, z)$ is a solution of equation* (1) *then the function*

$$w_x - \frac{1}{6}(x - a)\Delta w \tag{3}$$

is triharmonic, for any $(a, y, z) \in D$.

Proof. We show that

$$\Delta^3\left[w_x - \frac{1}{6}(x - a)\Delta w\right] = 0. \tag{4}$$

For this we use the linearity and omogenity of the operator Δ^3. We have successively

$$\Delta^3\left[w_x - \frac{1}{6}(x - a)\Delta w\right] = \Delta^3 w_x - \frac{1}{6}\Delta^3[(x - a)\Delta w] = \Delta^3 w_x -$$
$$- \frac{1}{6}\Delta^2[(x - a)\Delta^2 w + 2\Delta w_x] = \Delta^3 w_x - \frac{1}{6}\Delta[(x - a)\Delta^3 w + 2\Delta^2 w_x +$$
$$+ 2\Delta^2 w_x] = \Delta^3 w_x - \frac{1}{6}[(x - a)\Delta^4 w + 2\Delta^3 w_x + 4\Delta^3 w_x] = \Delta^3 w_x -$$
$$- \frac{1}{6}(x - a)\Delta^4 w - \Delta^3 w_x = 0,$$

because w is a solution of equation (1). Thus the Theorem is proved.

Also, we have the following

THEOREM 2. *If $w = w(x, y, z)$ is a solution of equation* (1) *then the function*

$$w_{xx} - \frac{1}{6}\Delta w - \frac{1}{3}(x - a)\Delta w_x + \frac{1}{24}(x - a)^2\Delta^2 w \tag{5}$$

is biharmonic, for any $(a, y, z) \in D$.

Proof. We must show that

$$\Delta^2\left[w_{xx} - \frac{1}{6}\Delta w - \frac{1}{3}(x - a)\Delta w_x + \frac{1}{24}(x - a)^2\Delta^2 w\right] = 0. \tag{6}$$

We have successively

$$\Delta^2\left[w_{xx} - \frac{1}{6}\Delta w - \frac{1}{3}(x - a)\Delta w_x + \frac{1}{24}(x - a)^2\Delta^2 w\right] = \Delta^2 w_{xx} - \frac{1}{6}\Delta^3 w -$$
$$- \frac{1}{3}\Delta[(x - a)\Delta^2 w_x + 2\Delta w_{xx}] + \frac{1}{24}\Delta[2\Delta^2 w + (x - a)^2\Delta^3 w + 4(x - a)\Delta^2 w_x] =$$
$$= \Delta^2 w_{xx} - \frac{1}{6}\Delta^3 w - \frac{1}{3}[(x - a)\Delta^3 w_x + 2\Delta^3 w_{xx} + 2\Delta^2 w_{xx}] + \frac{1}{24}\Delta[2\Delta^2 w +$$
$$+ (x - a)^2\Delta^3 w + 4(x - a)\Delta^2 w_x] = \Delta^2 w_{xx} - \frac{1}{6}\Delta^3 w - \frac{1}{3}[(x - a)\Delta^3 w_x +$$
$$+ 2\Delta^3 w_{xx} + 2\Delta^2 w_{xx}] + \frac{1}{24}[2\Delta^3 w + 2\Delta^3 w + (x - a)^2\Delta^4 w + 4(x - a)\Delta^3 w_x +$$
$$+ 4(x - a)\Delta^3 w_x + 8\Delta^2 w_{xx}] = \Delta^2 w_{xx} - \frac{1}{6}\Delta^3 w - \frac{1}{3}(x - a)\Delta^3 w_x -$$
$$- \frac{4}{3}\Delta^3 w_{xx} + \frac{1}{6}\Delta^3 w + \frac{1}{24}(x - a)^2\Delta^4 w + \frac{1}{3}(x - a)\Delta^3 w_x + \frac{1}{3}\Delta^3 w_{xx} = 0,$$

because w is a solution of equation (1), therefore the Theorem is proved.

Remark 1. The Theorem 2 may be demonstrated and by using the Theorem 1 from [2] for the triharmonic function $w_x - \frac{1}{6}(x - a)\Delta w$.

Analogous, it holds the following

THEOREM 3. *If $w = w(x, y, z)$ is a solution of equation* (1) *then the function*

$$w_{xxx} - \frac{1}{2}\Delta w_x - \frac{1}{2}(x - a)\Delta w_{xx} + \frac{1}{8}(x - a)\Delta^2 w + \frac{1}{8}(x - a)^2\Delta^2 w_x - \frac{1}{48}(x - a)^3\Delta^3 w \tag{7}$$

is harmonic, for any $(a, y, z) \in D$.

Proof. We must show that

$$\Delta\left[w_{xxx} - \frac{1}{2}\Delta w_x - \frac{1}{2}(x - a)\Delta w_{xx} + \frac{1}{8}(x - a)\Delta^2 w + \frac{1}{8}(x - a)^2\Delta^2 w_x - \frac{1}{48}(x - a)^3\Delta^3 w\right] = 0. \tag{8}$$

We have successively

$$\Delta\left[w_{xxx} - \frac{1}{2}\Delta w_x - \frac{1}{2}(x - a)\Delta w_{xx} + \frac{1}{8}(x - a)\Delta^2 w + \frac{1}{8}(x - a)^2\Delta^2 w_x -\right.$$
$$\left. - \frac{1}{48}(x - a)^3\Delta^3 w\right] = \Delta w_{xxx} - \frac{1}{2}\Delta^2 w_x - \frac{1}{2}[(x - a)\Delta^2 w_{xx} + 2\Delta w_{xxx}] +$$
$$+ \frac{1}{8}[(x - a)\Delta^3 w + 2\Delta^2 w_x] + \frac{1}{8}[2\Delta^2 w_x + (x - a)^2\Delta^3 w_x + 4(x - a)\Delta^2 w_{xx}] -$$
$$- \frac{1}{48}[6(x - a)\Delta^3 w + (x - a)^3\Delta^4 w + 6(x - a)^2\Delta^3 w_x] = \Delta w_{xxx} -$$
$$- \frac{1}{2}\Delta^2 w_x - \frac{1}{2}(x - a)\Delta^2 w_{xx} - \Delta w_{xxx} + \frac{1}{8}(x - a)\Delta^3 w + \frac{1}{4}\Delta^2 w_x +$$
$$+ \frac{1}{4}\Delta^2 w + \frac{1}{8}(x - a)^2\Delta^3 w_x + \frac{1}{2}(x - a)\Delta^2 w_{xx} - \frac{1}{8}(x - a)\Delta^3 w -$$
$$\frac{1}{48}(x - a)^3\Delta w - \frac{1}{8}(x - a)^2\Delta^3 w_x = 0,$$

because w is a solution of equation (1), therefore the Theorem is proved

Remark 2. The Theorem 3 may be established using and the Theorem from [2] for the triharmonic function $w_x - \frac{1}{6}(x - a)\Delta w$.

Also, it holds and the following

THEOREM 4. *If the function* $w = w(x, y, z)$ *is a solution of equation* (1) *then the function*

(9) $$\Delta^2 w_x - \frac{1}{2}(x-a)\Delta^3 w$$

is harmonic, for any $(a, y, z) \in D$.

Proof. We have successively

$$\Delta\left[\Delta^2 w_x - \frac{1}{2}(x-a)\Delta^3 w\right] = \Delta^3 w_x - \frac{1}{2}[(x-a)\Delta^4 w + 2\Delta^3 w_x] =$$

$$= \Delta^3 w_x - \frac{1}{2}(x-a)\Delta^4 w - \Delta^3 w_x = 0,$$

which shows that the function (9) is harmonic, and the Theorem is demonstrated.

3. In this section, by using maximum principles, we establish some estimates for the functions of theorems from previous section.

Thus, it holds the following

THEOREM 5. *Let* $w = w(x, y, z)$ *be a solution of equation* (1) *in the domain* $D \subset R^3$, *which have all partial derivatives until the* 6-*th order continuous and on the boundary* Γ *of domain* D. *Then, for any point* $(a, b, c) \in D$ *the inequality*

(10)
$$w_{xxx}(a, b, c) - \frac{1}{2}\Delta w_x(a, b, c) \leq \max_{\Gamma}\left\{w_{xxx} - \frac{1}{2}\Delta w_x - \frac{1}{2}(x-a)\Delta w_{xx} + \right.$$
$$\left. + \frac{1}{8}(x-a)\Delta^2 w + \frac{1}{8}(x-a)^2\Delta^2 w_x - \frac{1}{48}(x-a)^3\Delta^3 w\right\}$$

holds.

Proof. According to Theorem 3 the function (7) is harmonic in the domain D. So, we can apply to it the common maximum principle and the inequality (10) is obtained.

Proceeding analogous for the harmonic function (9) in Theorem 4 we have

THEOREM 6. *Under the assumptions of Theorem* 5 *the following inequality*

(11) $$\Delta^2 w_x(a, b, c) \leq \max_{\Gamma}\left\{\Delta^2 w_x - \frac{1}{2}(x-a)\Delta^3 w\right\}$$

holds.

For establishing a new result we introduce the following notations:

$$\nabla_3 w = (w_{xxx}, w_{yyy}, w_{zzz}),\ (\Delta w)' = (\Delta w_x, \Delta w_y, \Delta w_z),$$
$$(\Delta w)'' = (\Delta w_{xx}, \Delta w_{yy}, \Delta w_{zz}),\quad \mathbf{r}_1 = (x-a, y-b, z-c),$$
$$\mathbf{r}_1 \times (\Delta w)'' = ((x-a)\Delta w_{xx}, (y-b)\Delta w_{yy}, (z-c)\Delta w_{zz}),$$
(12) $$\mathbf{r}_1\Delta^2 w = ((x-a)\Delta^2 w, (y-b)\Delta^2 w, (z-c)\Delta^2 w),$$
$$\mathbf{r}_2 = ((x-a)^2, (y-b)^2, (z-c^2)),\ \mathbf{r}_3 = ((x-a)^3, (y-b)^3, (z-c)^3)$$
$$(\Delta^2 w)' = (\Delta^2 w_x, \Delta^2 w_y, \Delta^2 w_z),\ \mathbf{r}_2 \times (\Delta^2 w)' = ((x-a)^2\Delta^2 w_x,$$
$$(y-b)^2\Delta^2 w_y, (z-c)^2\Delta^2 w_z),\ \mathbf{r}_3\Delta^3 w = ((x-a)^3\Delta^3 w, (y-b)^3\Delta^3 w,$$
$$(z-c)^3\Delta^3 w).$$

The following theorem holds

THEOREM 7. *Under the assumptions of Theorem* 5 *we have the inequality*

(13)
$$\left|\nabla_3 w(a, b, c) - \frac{1}{2}(\Delta w(a, b, c))'\right| \leq \max_{\Gamma}\left|\nabla_3 w - \frac{1}{2}(\Delta w)' - \right.$$
$$\left. - \frac{1}{2}\mathbf{r}_1 \times (\Delta w)'' + \frac{1}{8}\mathbf{r}_1 \cdot \Delta^2 w + \frac{1}{8}\mathbf{r}_2 \times (\Delta^2 w)' - \frac{1}{48}\mathbf{r}_3\Delta^3 w\right|.$$

Proof. The vector

(14)
$$\nabla_3 w - \frac{1}{2}(\Delta w)' - \frac{1}{2}\mathbf{r}_1 \times (\Delta w)'' + \frac{1}{8}\mathbf{r}_1\Delta^2 w + \frac{1}{8}\mathbf{r}_2 \times (\Delta^2 w)' - \frac{1}{48}\mathbf{r}_3\Delta^3 w$$

has as components the following functions

$$w_{xxx} - \frac{1}{2}\Delta w_x - \frac{1}{2}(x-a)\Delta w_{xx} + \frac{1}{8}(x-a)\Delta^2 w + \frac{1}{8}(x-a)^2\Delta^2 w_x -$$
$$- \frac{1}{48}(x-a)^3\Delta^3 w,$$
$$w_{yyy} - \frac{1}{2}\Delta w_y - \frac{1}{2}(y-b)\Delta w_{yy} + \frac{1}{8}(y-b)\Delta^2 w + \frac{1}{8}(y-b)^2\Delta^2 w_y -$$
(15) $$- \frac{1}{48}(y-b)^3\Delta^3 w,$$
$$w_{zzz} - \frac{1}{2}\Delta w_z - \frac{1}{2}(z-c)\Delta w_{zz} + \frac{1}{8}(z-c)\Delta w + \frac{1}{8}(z-c)^2\Delta^2 w_z -$$
$$- \frac{1}{48}(z-c)^3\Delta^3 w,$$

which are harmonic functions, in accordance with the Theorem 3.

We know (see [1]) that the square of an harmonic function u is subharmonic ($\Delta u \geqslant 0$). Then the square of modulus of the vector (14) is a sum of subharmonic functions, which is a subharmonic function. Now, we apply the maximum principle for subharmonic functions, and we deduce that the inequality (13) holds, and thus the Theorem is proved.

We finish this section with the following

THEOREM 8. *Under the assumptions of Theorem* 5 *the inequality*

$$w_{xx}(a,\ b,\ c) - \frac{1}{6}\Delta w(a,\ b,\ c) \leqslant \max_{\Gamma}\left\{w_{xx} - \frac{1}{6}\Delta w - \frac{1}{3}(x-a)\Delta w_x + \right.$$

$$(16)\quad + \frac{1}{24}(x-a)^2\Delta^2 w - 2\mathbf{r}_1\nabla\left[w_{xx} - \frac{1}{6}\Delta w - \frac{1}{3}(x-a)\Delta w_x + \right.$$

$$\left. + \frac{1}{24}(x-a)^2\Delta^2 w\right] + \frac{1}{2}\mathbf{r}_1^2\Delta\left[w_{xx} - \frac{1}{6}\Delta w - \frac{1}{3}(x-a)\Delta w_x + \right.$$

$$\left.\left. + \frac{1}{24}(x-a)^2\Delta^2 w\right]\right\},$$

holds.

Proof. The function

$$w_{xx} - \frac{1}{6}\Delta w - \frac{1}{3}(x-a)\Delta w + \frac{1}{24}(x-a)^2\Delta^2 w$$

is the biharmonic function (5) of Theorem 2.

We know (see [1]) that if u is a 3-dimensional biharmonic function then

$$(17)\qquad u - 2\mathbf{r}_1\nabla u + \frac{1}{2}\mathbf{r}_1^2\Delta u$$

is a 3-dimensional harmonic function.

Thus for the function (17), where u is the biharmonic function (5), we can apply the common maximum principle, which leads us to the inequality (16). Thus the Theorem is proved.

REFERENCES

[1] Duffin, R. J., *The maximum principle and biharmonic functions* J. Math. Anal. Appl. **3**, 399–405 (1961).

[2] Mureşan, A., *Application of maximum principles to triharmonic functions.* Mathematica, **18**(41), 1, 103–109 (1976).

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