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ACADÉMIE DE LA RÉPUBLIQUE SOCIALISTE DE ROUMANIE FILIALE DE CLUJ-NAPOCA

MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION

TOME 9

Nº 1

1980

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ANALYSE NUMERIQUE

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L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION Tome 9, No 1, 1980, pp. 5-10 i) An impelacible with west diagonal dominance undrix ii) A positive definite mature of the control waters

SOME THEORETICAL AND COMPUTATIONAL RESULTS CONCERNING THE ACCELERATED OVERRELAXATION (AOR) METHOD

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G. AVDELAS, A. HAD JIDIMOS and A. YEYIOS

(Ioannina, Greece)

of the AOR method (P. J. O) are viventile terms of the circultraffices w of the = v II . No = z orony a _1. Introduction research and vd botom gos

Proof. By relationship (2.7) (HABITIMESS, 1978) The rigenvulnes A.

Very recently HADJIDIMOS (1978) has introduced a new iterative method for the numerical solution of a linear system Ax = b, where A is an $n \times n$ known matrix, x an unknown n-dimensional vector and b a known vector of the same dimension. By splitting A into the sum $D - A_L -A_{U}$, where D is the diagonal part of A and A_{U} and A_{U} the strictly lower and upper triangular parts of A multiplied by -1 and assuming that det $(D) \neq 0$, the corresponding AOR scheme has the following form: as was given by man manus (1978)

$$(1.1) (I - rL)x^{(n+1)} = [(1 - \omega)I + (\omega - r)L + \omega U]x^{(n)} + \omega c | n = 0, 1, 2, \dots$$

where $L = D^{-1}A_L$, $U = D^{-1}A_U$, $c = D^{-1}b$, I is the unit matrix of order n, r is the acceleration parameter, $\omega \neq 0$ is the overrelaxation parameter and $x^{(0)}$ arbitrary. The iterative matrix of scheme (1.1) is given by

$$L_{r,\,\omega}=(I-rL)^{-1}[(1-\omega)I+(\omega-r)L+\omega U].$$

This new two-parameter method is obviously a generalization of the SOR method (since for $r = \omega$ AOR coincides with SOR) and it should be noted that for $r \neq 0$ the AOR method is the Extrapolated SOR (ESOR) method with overrelaxation parameter r and extrapolation one $s = \omega/r$. Without loss of generality we shall assume throughout this paper that $A \equiv I - L - U$ since by premultiplication by D^{-1} the new coefficient matrix A of the original system will have this form.

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The purpose of this paper is to present some further basic results concerning the AOR method when the matrix A is

- i) An irreducible with weak diagonal dominance matrix
- ii) A positive definite matrix
- iii) An L-matrix and
- iv) An M-matrix

and also to show by mumerical examples its superiority compared with the sor method.

STHURBER LANGERA 2. A Basic Theorem Here we state and prove a basic theorem which helps us to extend and improve the results of HAD JIDIMOS (1978).

THEOREM 1. Let A be a nonsingular matrix. If the SOR method with overrelaxation parameter r converges for $0 < \alpha \le r \le \beta < 2$, then the AOR method converges for all r and s such that $\alpha \leq r \leq \beta$ and $0 < s \leq 1$ where $s = \omega/t$.

Proof. By relationship (2.7) (HADJIDIMOS, 1978) the eigenvalues λ_i of the AOR method $(r \neq 0)$ are given in terms of the eigenvalues v_i of the sor metod by the expressions $\lambda_i = sv_i + 1 - s$ where $s = \omega/r$. If $v_i =$ solve interest by the expressions $\alpha_i = \alpha_i$ with $0 \le \rho < 1$, then for $\alpha \le r \le \beta$ and $0 < s \le 1$ we have that $|\lambda_i|^2 = s^2 \rho^2 + 2s\rho(1-s)\cos\theta + (1-s)^2 \le (s\rho + 1 - s)^2 = 1$

$$|\lambda_i|^2 = s^2 \rho^2 + 2s \rho (1-s) \cos \theta + (1-s)^2 \le (s\rho + 1-s)^2 =$$

$$= [1-s(1-\rho)]^2 < 1$$
and the sore matrix 1.

and the AOR method converges for all $\alpha \le r \le \beta$ and $0 < s \le 1$. In what follows we examine the extensions concerning the theory of AOR method as was given by HADJIDIMOS (1978) in view of the theorem which has just been proved.

3. Irreducible Matrices with Weak Diagonal Dominance

THEOREM 1. Let A be an irreducible matrix with weak diagonal dominance. Then the AOR method ($r \neq 0$) converges for all $0 < r \leq 1$ and $0 < r \leq 1$ $(x_0 = (x_0 + x_0) + (x_0 = x_0)) + (x_0 = x_0) + (x_0 =$ $< s \le 1$.

Proof. In all basic books (see VARGA, 1962; WACHSPRESS, 1966; YOUNG, 1971) a classical theorem concerning the sor method is presented stating that the sor method converges for $0 < r \le 1$. Thus by Theorem 2.1 above we conclude that the AOR method converges for all $0 < r \le 1$ and $0 < s \le 1$. (i.e. $0 < \omega \le r \le 1$). This is a corollary to the theorem of section 3 (HADJIDIMOS, 1978) according to which the AOR method converges for all $0 \le r \le 1$ and $0 < \omega \le 1$, not side event liew matrix lamining

4. Positive Definite Matrices

THEOREM 1. Let A be a positive definite (real) matrix. Then the AOR method converges for $0 < \omega \le r \le 2$ ($\omega \ne 2$).

Proof. Since A = I - L - U we have $L = U^T$ and also that $y^H A y > 0$ > 0 for every complex n-dimensional vector v with v^H denoting the conjugate transpose of y. If λ is an eigenvalue of $L_{r,\omega}$, then for some $\nu \neq 0$ we shall have $L_{r,\omega} v = \lambda v$ and hence

$$(I-rL)^{-1}[(1-\omega)I+(\omega-r)L+\omega U]$$
v $=\lambda$ v, or $[(1-\omega)I+(\omega-r)L+\omega U]$ v $=\lambda(I-rL)$ v.

Multiplying both sides on the left by v^H and solving for λ , assuming for

the moment that
$$v^H(I-rL)v \neq 0$$
, we obtain
$$\lambda = \frac{(1-\omega)v^Hv + (\omega-r)v^HLv + \omega v^HUv}{v^H(I-rL)v} = \frac{(1-\omega)v^Hv + (\omega-r)v^HLv + \omega v^HUv}{v^Hv - rv^HLv}.$$

Since $L = U^T$ we have

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$$\nabla^H U_{\mathsf{V}} = \mathsf{V}^H L^T \mathsf{V} = (\mathsf{V}^T L \mathsf{V}^*)^T = \mathsf{V}^T L \mathsf{V}^* = (\mathsf{V}^H L \mathsf{V})^*$$

where v* is the complex conjugate matrix of v. Thus if we let body and the part to bound be will be and a way of the

(4.1)
$$\frac{v^{H}L_{v}}{v^{H_{v}}} = z, \text{ then}$$

(4.2)
$$z^* = \left(\frac{\sqrt{HL_V}}{\sqrt{H_V}}\right)^* = \frac{(\sqrt{HL_V})^*}{\sqrt{H_V}} = \frac{\sqrt{HU_V}}{\sqrt{H_V}} \text{ and hence}$$

$$\lambda = \frac{1 - \omega + (\omega - r)z + \omega z^*}{1 - rz}.$$

If we put $z = \alpha + \beta i$ where α and β are real we have

$$|\lambda| = \left| rac{(1 - rlpha + 2\omegalpha - \omega) - reta i}{1 - rlpha - reta i}
ight|.$$

To prove the validity of the theorem it is sufficient to show that $|\lambda| < 1$ for $0 < \omega \le r \le 2$ ($\omega \ne 2$) or equivalently that

$$\omega^2(2\alpha-1)^2+2\omega(1-r\alpha)(2\alpha-1)<0.$$

Since $\omega > 0$ we must show that

(4.3)
$$\omega(2\alpha - 1)^2 + 2(1 - r\alpha)(2\alpha - 1) < 0.$$

Using (4.1) and (4.2) we have that

$$z + z^* = 2\alpha = \frac{v^H L v}{v^H v} + \frac{v^H U v}{v^H v} = \frac{v^H (L + U) v}{v^H v} = \frac{v^H (I - A) v}{v^H v} = 1 - \frac{v^H A v}{v^H v} < 1$$

since A is positive definite. This gives that $\alpha < 1/2$. Relationship (4.3) is now equivalent to I = I = I we have I = I = I and I = I

$$(4.4) \qquad \qquad \omega(2\alpha - 1) + 2(1 - r\alpha) > 0$$

which, we observe, is satisfied for $r = \omega$, since $2 - \omega > 0$. For r = 2 (4.4) gives $(1-2\alpha)(2-\omega)>0$ which is also valid. For $r\neq \omega$, 2 we have to distinguish cases. Thus if $\alpha \le 0$ (4.4) becomes $-2\alpha(r-\omega) + 2 - \omega > 0$ which is readily seen to hold. If $\alpha > 0$ then we have

$$-2\alpha(r-\omega)+2-\omega>-2\alpha(r-\omega)+r-\omega=(r-\omega)(1-2\alpha)>0.$$

Thus $\left|\lambda\right|<1$ and the convergence follows. It remains to be proved that $v^H(I-rL)v\neq 0$. For this we assume that $v^H(I-rL)v=0$. Then it will be

$$\frac{v^H(I-rL)v}{v^Hv}=0$$
 giving that $1-r\frac{v^HLv}{v^Hv}=0$, or $1-rz=0$,

which implies that $1 - r\alpha = 0$ and $r\beta = 0$. Since $r \neq 0$ we have that $\beta = 0$ and $\alpha = 1/r$. But $1/r \ge 1/2$ or equivalently $2\alpha \ge 1$ which is not possible because $2\alpha < 1$.

Remark. By theorems 3.6 (young, 1971, p. 113) and 2.1 of this paper we can show the convergence of the AOR method for 0 < r < 2and $0 < s \le 1$, where $s = \omega/r$. This, however does not include the case r=2. So the theorem 4.1 of the present section is more general.

5. L-and M-Matrices THEOREM 1. Let A be an L-matrix. Then the AOR method converges for all $0 < r \le 1$ and $0 < s \le 1$, where $s = \omega/r$ if and only if the Jacobi method converges $(\rho(B) < 1, B \equiv L + U)$.

Proof. If $\rho(B) < 1$ then by theorem 5.1 (a) (YOUNG, 1971, p. 120) we have that the sor method converges for $0 < r \le 1$ and therefore by theorem 2.1 of this paper the AOR method converges for $0 < r \le 1$ and $0 < s \le 1$. Conversely, if the AOR method converges for $0 < r \le 1$ and $0 < s \le 1$, then for s = 1 we have that the sor method converges for $0 < r \le 1$. Therefore theorem 5.1 (a) (young, 1971, p. 120) implies that $\rho(B) < 1$.

If we combine the theorem above with the theorem of section 4 (HADJIDIMOS, 1978) we conclude the following more general theorem for L-matrices.

THEOREM 2. Let A be an L-matrix. Then the AOR method converges for al $0 \le r \le 1$ and $0 < \omega \le 1$ if and only if the Jacobi method coverges $(\rho(B) < 1)$.

THEOREM 3. Let A be an L-matrix. Then each of the following statements is equivalent to the other two. S1. The Jacobi method converges. S2. The SOR method converges for $0 < r \le 1$. S3 The AOR method converges for $0 \le r \le 1$ and $0 < \omega \le 1$.

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Proof. From $\overline{|S1|}$ we can easily go to $\overline{|S2|}$ and $\overline{|S3|}$ by using theorem 5.2 above. By theorems 5.1 (a) (YOUNG, p. 120) and 5.2 of this paper |S2| implies $\overline{[S1]}$ and $\overline{[S3]}$. Finally $\overline{[S3]}$ implies $\overline{[S1]}$ for r=0, $\omega=1$ and $\overline{[S2]}$ for $\omega=r$. THEOREM 4. Let A be an L-matrix. Then, the AOR method converges for $0 < r < \frac{2}{1 + \rho(B)}$ and $0 < s \le 1$ if and only if A is an M-matrix.

Proof. If A is an M-matrix then by theorem 7.2 (YOUNG, p.43) we have $\rho(B) < 1$ and by theorem 5.9 (young, p. 126) we conclude that the sor method converges for $0 < r < \frac{2}{1+\rho(B)}$. Hence by theorem 2.1 of this paper the AOR method converges for $0 < r < \frac{2}{1 + \rho(B)}$ and $0 < s \le 1$. Conversely, if the AOR method converges for $0 < r < \frac{2}{1 + \rho(B)}$ and $0 < s \le 1$ this implies that the SOR method (s = 1) converges for $0 < r < \frac{2}{1 + \rho(B)}$. If we assume that $\rho(B) \ge 1$ then $\frac{2}{1+\rho(B)} \le 1$ and by theorem 5.1 (c) (YOUNG, p. 121) we have that the sor method does not converge for any value of rsuch that $0 < r \le 1$ which is not true. Thus we have $\rho(B) < 1$ and therefore A is an M-matrix according to theorem 7.2. (YOUNG, p. 43).

THEOREM 5. Let A be an M-matrix. Then the AOR method converges for $0 \le r \le 1$ and $0 < \omega \le 1$ and also for $1 < r < \frac{2}{1 + \rho(B)}$ and $0 < \omega \le r$.

Proof: The proof follows by Theorems 5.2 and 5.4. ACCIPATION Should be used in The Maccion the son mathal, whenever the

6. Numerical examples

The following examples ensure the validity of all theorems presented previously and in addition show the asymptotical superiority of the AOR method, compared with the sor one. For this, we worked out specific examples and found out the optimum spectral radii of the corresponding iterative matrices in all cases considered. We have, to the parameters r and ω all the values 0 (0.01) 2 and 0.01 (0.01) 2 respectively and independently of each other. The upper bound 2 was selected because all theorems of this paper give only sufficient conditions for the convergence of the AOR method and not necessary ones as well while in addition to that the theorem of Kahan states that if the SOR method converges, then it must be $0 < \omega < 2$.

Using the UNIVAC 1106 Computer of the University of Salonica and with a maximum permissible relative error $E = 10^{-4}$, in finding the spectral radii, we have found that:

i) In the case where A is the following irreducible with weak diagonal

dominance matrix

$$A = \begin{bmatrix} 1 & 0.5 & -0.25 \\ 0.6 & 1 & 0.2 \\ -0.5 & 0.3 & 1 \end{bmatrix},$$

the optimum spectral radius of AOR method has been found for r = 1.26, $\omega = 1.18$ and is $\rho_{\rm opt}(L_{\rm r,\,\omega}) \approx 0.272$ while the optimum spectral radius of sor method is given for $\omega = 1.24$ and is $\rho_{\rm opt}(L_{\omega,\omega}) \approx 0.327$.

ii) In the case where A is the following positive definite (real) matrix

$$A = egin{bmatrix} 1 & 0.4 & 0.4 \ 0.4 & 1 & 0.6 \ 0.4 & 0.6 & 1 \end{bmatrix}$$
 ,

it has been found that $\rho_{\rm opt}(L_{r,\,\omega})\approx 0.196$ for r=1.03 and $\omega=1.23$ while $\rho_{\rm opt}(L_{\omega, \omega}) \approx 0.282$ for $\omega = 1.08$.

iii) In the case where A is the following L-matrix with $\rho(B) < 1$, that

is, A is an M-matrix

$$A = \begin{bmatrix} 1 & 0 & -0.5 & 0 \\ 0 & 1 & -0.5 \\ -0.5 & 0 & 1 \end{bmatrix}, \quad 13$$

we have $\rho_{\rm opt}(L_{r,\,\omega})\approx 0.302$ for $r=1.18,\,\omega=0.90$ and $\rho_{\rm opt}(L_{\omega,\,\omega})\approx 0.354$ for $\omega=1.00$.

7. Final remarks

As has been seen from the numerical examples we gave in the previous section, it is always $\rho_{\rm opt}(L_{r,\,\omega})<\rho_{\rm opt}(L_{\omega,\,\omega})$, that is, the AOR method converges faster than the corresponding son one. This simply suggests that the AOR method should be used in the place of the sor method, whenever the latter is used.

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Received 3. IX. 1979

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