MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 9, N° 1, 1980, pp. 93—99

THE EXTENSION OF STARSHAPED BOUNDED LIPSCHITZ FUNCTIONS

by

C. MUSTĂȚA

1. Let X be a normed space and Y a nonvoid subset of X. The set Y is called *starshaped* (with respect to $\theta \in X$) if $\alpha y \in Y$ for all $\alpha \in [0, 1]$ and $y \in Y$. A real function f defined on Y is called *starshaped* if

$$f(\alpha y) \leq \alpha f(y), \tag{1.1}$$

for all $\alpha \in [0, 1]$ and $y \in Y$. From (1.1) follows $f(\theta) \leq 0$. In the following we consider only starshaped functions vanishing at θ. Y will denote always a starshaped set.

A function $f: Y \to R$ is called Lipschitz on Y if there exists $K \geq 0$ such that

$$|f(x) - f(y)| \leq K \cdot ||x - y||, \tag{1.2}$$

for all $x, y \in Y$. Denote by $\mathrm{Lip}_0 Y$ ($\mathrm{Lip}_0 X$) the space of all Lipschitz functions on Y (respectively on X) vanishing at $\theta \in Y \subset X$ ([5]) and by $\mathrm{BLip}_0 Y$ ($\mathrm{BLip}_0 X$) their subspaces formed of bounded functions on Y (respectively on X). Denote also

$$\mathrm{BS}_Y = \{f : f \in \mathrm{BLip}_0 Y,\ f \text{ is starshaped}\}, \tag{1.3}$$

$$\mathrm{BS}_X = \{F : F \in \mathrm{BLip}_0 X,\ F \text{ is starshaped}\}. \tag{1.4}$$

The sets BS_Y and BS_X are convex cones in $BLip_0Y$ and in $BLip_0X$, respectively, i.e. $f+g$ and λf are in BS_Y (in BS_X) for all f, g in BS_Y (in BS_X) and $\lambda \geq 0$.

For $f \in BS_Y$ ($f \in BLip_0Y$) put

$$\|f\|_d = \sup\{|f(y_1) - f(y_2)|/\|y_1 - y_2\| : y_1, y_2 \in Y, \ y_1 \neq y_2\} \tag{1.5}$$

the *Lipschitz norm of f*, and

$$\|f\|_\infty = \sup\{|f(y)| : y \in Y\}, \tag{1.6}$$

the *uniform norm of f*.

For $F \in BLip_0X$, the Lipschitz and the uniform norms are defined similarly.

2. It is well known (see e.g. [4]) that a Lipschitz function f defined on a nonvoid subset Y of a metric space X has a norm preserving Lipschitz extension F on X, i.e. $F|_Y = f$ and $\|F\|_d = \|f\|_d$. In [1] it was shown that if Y is a nonvoid convex subset of a normed space X, then every convex Lipschitz function on Y has a convex norm preserving Lipschitz extension F on X. In a similar result was established for starshaped Lipschitz function [7].

This Note is concerned with the problem of extension of bounded starshaped Lipschitz functions, i.e. for $f \in BS_Y$ find $F \in BS_X$ such that $F|_Y = f$, $\|F\|_d = \|f\|_d$ and $\|F\|_\infty = \|f\|_\infty$. The function F is called briefly an *extension* of f. We consider the problem of existence and unicity of such an extension.

Remark, that similar problem for bounded convex Lipschitz functions has a trivial answer: if Y is a convex subset of X, such that $\theta \in Y$, then the nul function is the only bounded convex (Lipschitz) function on Y which has a bounded convex Lipschitz extension on X. This follows from the fact that the constant functions are only bounded convex functions on X. Indeed, if $F : X \to R$ is a nonconstant convex function, then there exist two points $x_0, x_1 \in X$ such that $F(x_0) \neq F(x_1)$, say $F(x_0) < F(x_1)$. But then, since the function $\varphi : (0, \infty) \to R$, defined by

$$\varphi(t) = \frac{F(x_0 + t(x_1 - x_0)) - F(x_0)}{t}, \quad t > 0$$

is nondecreasing (see [2] p. 17), it follows that

$$\frac{F(x_0 + t(x_1 - x_0)) - F(x_0)}{t} \geq F(x_1) - F(x_0) > 0,$$

so that $F(x_0 + t(x_1 - x_0)) \geq F(x_0) + t[F(x_1) - F(x_0)]$, for all $t \geq 1$, which shows that the function F is unbounded.

2a. The existence of extension. In Theorem 1 below will be shown that under some suplementary hypotheses on the function $f \in BS_Y$ there exists an extension $F \in BS_X$ of f.

Firstly, we prove two lemmas.

LEMMA 1. *Let X be a real normed space and f a starshaped function on X. Then, for every $x \in X$, $x \neq \theta$, the function $\Psi : (0, \infty) \to R$, defined by*

$$\Psi(t) = f(tx)/t, \quad t > 0 \tag{2.1}$$

is nondecreasing.

Proof. For $0 < t_1 < t_2$ and a fixed $x \in X$, $x \neq \theta$, we have

$$\frac{f(t_1x)}{t_1} = \frac{f((t_1/t_2)t_2x)}{t_1} \leq \frac{(t_1/t_2)f(t_2x)}{t_1} = \frac{f(t_2x)}{t_2}.$$

Now, for a function $f : X \to R$ define, as usually, the *epigraph* of f, by

$$\text{epi}\, f = \{(x, \alpha) \in X \times R : f(x) \leq \alpha\}. \tag{2.2}$$

LEMMA 2. *A function $f : X \to R$, $f(\theta) = 0$, is starshaped if and only if its epigraph is starshaped.*

Proof. If f is starshaped, $f(\theta) = 0$, and $(x, \alpha) \in \text{epi}\, f$, then for every $\lambda \in [0, 1]$, $f(\lambda x) \leq \lambda f(x) \leq \lambda\alpha$, so that $\lambda(x, \alpha) = (\lambda x, \lambda\alpha) \in \text{epi}\, f$. Conversely, if epi f is starshaped then $(x, f(x)) \in \text{epi}\, f$ implies $(\lambda x, \lambda f(x)) \in \text{epi}\, f$, i.e. $f(\lambda x) \leq \lambda f(x)$, for all $\lambda \in [0, 1]$.

THEOREM 1. *Let X be a normed space, Y a starshaped subset of X, $\theta \in Y$, and $f \in BS_Y$. Then, there exists $F \in BS_X$ such that*

(i) $F|_Y = f$,

(ii) $\|F\|_d = \|f\|_d$,

(iii) $\|F\|_\infty = \|f\|_\infty$,

if and only if $f(y) \leq 0$, for all $y \in Y$.

Proof. Let $f \in BS_Y$ and suppose $f \leq 0$ on Y. Define $G : X \to R$ by

$$G(x) = \inf_{y \in Y} [f(y) + \|f\|_d \|x - y\|]. \tag{2.3}$$

The function G defined by (2.3) is starshaped and satisfies $G|_Y = f$, $\|G\|_d = \|f\|_d$ (see [7]).

Let

$$F(x) = \begin{cases} 0 & \text{if } G(x) > 0, \\ G(x) & \text{if } G(x) \leq 0. \end{cases} \tag{2.4}$$

Since $G|_Y = f \le 0$, it follows $F|_Y = f$ and $||F||_d = ||f||_d$. Obviously $||F||_\infty \ge ||f||_\infty$. If $x \in X$ is such that $G(x) \le 0$, then

$$0 \ge F(x) = G(x) \ge f(y) + ||f||_d \cdot ||x - y|| \ge f(y),$$

so that $0 \le -F(x) \le -f(y)$, for all $y \in Y$. Therefore $||F||_\infty = \inf_{x \in X} (-F(x)) \le \le ||f||_\infty$ and $||F||_\infty = ||f||_\infty$. Since the epigraph of F is starshaped, by Lemma 2, F is starshaped. Consequently, F is the required extension of f.

Suppose now, that there exists $y_0 \in Y$ such that $f(y_0) > 0$, and let F be a starshaped extension of f. By Lemma 1

$$0 < f(y_0) = F(y_0) \le (F(ty_0)/t),$$

so that $F(ty_0) \ge tf(y_0)$, for all $t \ge 1$, which shows that the function F is unbounded. Therefore f has no bounded strashaped extension, which ends the proof of Theorem 1.

Let

$$\mathrm{BS}_Y^- = \{f \in \mathrm{BS}_Y : f \le 0\},$$
$$\mathrm{BS}_X^- = \{F \in \mathrm{BS}_X : F \le 0\}. \tag{2.5}$$

By Theorem 1 follows:

COROLLARY 1. *Every function $f \in \mathrm{BS}_Y^-$ has an extension $F \in \mathrm{BS}_X^-$.*

2b. The unicity of extension. By Theorem 1 and Corollary 1, every nonpositive bounded starshaped function, defined on a starshaped subset Y of a normed space X, has a nonpositive bounded starshaped extension to whole X. Furthemore, these are the only bounded starshaped function on Y admitting bounded starshaped Lipschitz extension on X.

Equiped with the norms

$$||f||_Y = \max\,(||f||_d,\ ||f||_\infty),\quad f \in \mathrm{BLip}_0\, Y$$
$$||F||_X = \max\,(||F||_d,\ ||F||_\infty),\quad F \in \mathrm{BLip}_0\, X \tag{2.6}$$

$\mathrm{BLip}_0 Y$ and $\mathrm{BLip}_0 X$ become Banach spaces (see [3]). Let $H = \mathrm{BS}_X^- - - \mathrm{BS}_X^-$ the subspace of $\mathrm{BLip}_0 X$ generated by the convex cone BS_X^- and

$$Y^\perp = \{g \in H : g|_Y = 0\},$$

the anihilator of the set Y in H. Obviously, $Y^\perp$ is a subspace of H. A subset Z of a normed space X is called *proximinal* for $W \subset X$ if for every $f \in W$ there exists $g_0 \in Z$, such that

$$||f - g_0|| = d(f, Z) = \inf\{||f - g|| : g \in Z\}. \tag{2.7}$$

If for every $f \in W$ the element $g_0 \in Z$ satisfying (2.7) is unique then the set Z is called *Chebyshevian for* W. An element $g_0 \in Z$ satisfying (2.7) is called an *element of best approximation of f* by elements of Z.

THEOREM 2. *$Y^\perp$ is a Chebyshevian subsapce for BS_X^- if and only if, every $f \in \mathrm{BS}_Y^-$ has a unique (preserving the uniform and Lipschitz norms) extension F in BS_X^-.*

Proof. Follows from Theorem 1 in [6].

Remark. Observe that Theorem 2 remains true if the spaces $\mathrm{BLip}_0 Y$ and $\mathrm{BLip}_0 X$ are equiped with the norms

$$||f||_1 = ||f||_d + ||f||_\infty,\ \textit{for}\ f \in \mathrm{BLip}_0 Y\ \textit{(respectively}\ \mathrm{BLip}_0 X) \tag{2.8}$$

Theorem 2 is analogous with a theorem of PHELPS [8], in the linear case.

3. Now, we try to find conditions on the function f ensuring the unicity of the extension.

Consider, firstly, the case $X = R$ with the usual norm $|\cdot|$ (the absolute value).

THEOREM 3. *Let $Y = [a, b] \subset R$, $a < 0 < b$. A function $f \in \mathrm{BS}_Y^-$ has a unique extension $F \in \mathrm{BS}_R^-$ if and only if $f(a) = f(b) = 0$.*

Proof. Suppose that $f \in \mathrm{BS}_Y^-$ has two distinct extensions F_1, F_2 in BS_R^-. Let $x \in R \setminus [a, b]$ be such that $F_1(x) \ne F_2(x)$, say $F_1(x) < F_2(x) \le 0$. Suppose $x > b$. The function F_1 being starshaped it follows

$$F_1(\lambda x) \le \lambda F_1(x) < 0,$$

for all $\lambda \in (0, 1]$. In particular, since $b = \lambda_b x$ for $\lambda_b \in (0, 1)$, it follows $f(b) = F_1(b) < 0$. If $x < a$, then $a = \lambda_a x$, for a $\lambda_a \in (0, 1)$, and similarly, $f(a) = F_1(a) < 0$.

Conversely, we shall show that if $f \in \mathrm{BS}_Y^-$ is such that $f(a) < 0$ or $f(b) < 0$, then f has at least two distinct extensions F_1 and F_2 in BS_R^-. If $f(b) < 0$, then

$$F_1(x) = \begin{cases} f(x), & x \in [0, b] \\ f(b) + ||f||_d(x - b), & x \in (b,\ b - f(b)(||f||_d)^{-1}] \\ 0, & x \in (-\infty, a) \cup (b - f(b)\cdot(||f||_d)^{-1}, +\infty) \end{cases}$$

and

$$F_2(x) = \begin{cases} f(x) & ,\ x \in [0, b] \\ (f(b)/b)x, & x \in (b, (-||f||_\infty/f(b))b) \\ -||f||_\infty & ,\ x \in [-(||f||_\infty/f(b))b, +\infty) \\ 0 & ,\ x \in (-\infty, a) \end{cases}$$

are two distinct extensions of f, i.e. $F_1 \ne F_2$, $F_1|_Y = F_2|_Y = f$, $||F_1||_d = = ||F_2||_d = ||f||_d$, $||F_1||_\infty = ||F_2||_\infty = ||f||_\infty$.

$F_1(x) \neq F_2(x)$ for all $x > \max \{b - f(b) \cdot (\|f\|_d)^{-1};\ -\|f\|_\infty \cdot (f(b))^{-1}\}$.

If $f(a) < 0$, then two distinct extensions F_1, F_2, may be given, in a similar way.

Remark. The hypothesis $a < 0 < b$ in Theorem 3 is essential as it is shown by the following example. Take $Y = [0, b]$, $b > 0$ or $Y = [a, 0]$, $a < 0$. Then every $f \in BS_Y^-$ has an infinite set of extensions in BS_R^-. For exemple, if $Y = [0, b]$ and $f \in BS_Y^-$ is such that $f(0) = 0$, $f(b) = 0$, then

$$F_\lambda(x) = \begin{cases} f(x) & , \quad x \in [0, b], \\ 0 & , \quad x \in (0, +\infty) \\ \|f\|_d x & , \quad x \in (-\lambda\|f\|_\infty \cdot (\|f\|_d)^{-1}, 0) \\ -\lambda\|f\|_\infty, & x \in (-\infty, -\lambda\|f\|_\infty(\|f\|_d)^{-1}) \end{cases}$$

is an extension of f for every $\lambda \in [0, 1]$.

Consider now the general case. For $x \in X$, $x \neq \theta$, the ray $\overrightarrow{\theta x}$ is defined by

$$\overrightarrow{\theta x} = \{\alpha x : \alpha \geq 0\}.$$

If the set Y is starshaped and $y \in Y$, $y \neq \theta$, then $\overrightarrow{\theta y} \subset Y$ or $\overrightarrow{\theta y} \cap Y$ is a segment. In the second case put

$$\alpha_y = \sup \{\alpha : \alpha y \in Y\},$$

and $z_y = \alpha_y \cdot y$. The set $\{z_y : y \in Y\}$ is called *the algebric starshapes boundary* of Y and is denoted by $\mathrm{Fr}^s Y$.

Evidently, every $z \in \mathrm{Fr}^s Y$ is a limit point of Y, i.e. $Y \cup \mathrm{Fr}^s Y \subset \overline{Y}$. Since every $f \in BS_Y$ is uniformly continuous (as Lipschitz) it can be uniquely extended to $Y \cup \mathrm{Fr}^s Y$. Therefore with no restrition of generality, we can suppose $\mathrm{Fr}^s Y \subset Y$.

THEOREM 4. *Let Y be an absorbing starshaped subset of the normed space X, such that. $\mathrm{Fr}^s Y \subset Y$. If $f \in BS_Y^-$ is such that $f(z) = 0$, for all $z \in \mathrm{Fr}^s Y$, then f has a unique extension $F \in BS_X^-$.*

Proof. Suppose $f \in BS_Y^-$, $f(z) = 0$, for all $z \in \mathrm{Fr}^s Y$, and suppose that f has two distinct extensions F_1, F_2 in BS_X^-. Let $x \in X \setminus Y$ be such that $F_1(x) \neq F_2(x)$, say $F_1(x) < F_2(x) \leq 0$. The set Y being absorbing and starshaped, there exists $\lambda > 0$ such that $\lambda x \in \mathrm{Fr}^s Y$. But then, one obtains the contradiction

$$0 = f(\lambda x) = F_1(\lambda x) \leq \lambda F_1(x) < 0.$$

Theorem 4 is proved.

BIBLIOGRAPHY

[1] Cobzaş, S., Muştăţa, C., *Norm Preserving Extension of Convex Lipschitz Functions*, J. Approx. Theory **24** (1978), 555–564.

[2] Holmes, R. B., *A Course on Optimisation and Best Approximation*, Lectures Notes in Math. No. 257, Springer Verlag, Berlin-Heidelberg-New York, 1972.

[3] Johnson, J. A., *Banach Spaces of Lipschitz Functions and Vector-Valued Lipschitz Functions*, Trans. Amer. Math. Soc. **148** (1970), 147–169.

[4] McShane, E. J., *Extension of Range of Functions*, Bull. Amer. Math. Soc. **40** (1934), 837–842.

[5] Muştăţa, C., *Best Approximation and Unique Extension of Lipschitz Functions*, J. Approx. Theory **19** (1977), 222–230.

[6] Muştăţa, C., *A Characterisation of Chebyshevian Subspace of Y*$^{\perp}$*-Type*, Mathematica — Revue Anal. Num. Teor. Approx., L'Analyse Num. Teor. Approx. **6**, 1 (1977), 51–56.

[7] Muştăţa, C., *Norm Preserving Extension of Starshaped Lipschitz Functions*, Mathematica **19** (**42**) 2 (1977), 183–187.

[8] Phelps, R. R., *Uniqueness of Hahn-Banach Extension and Unique Best Approximation*, Trans. Amer. Math. Soc. **95** (1960), 238–255.

Received. 21.XII.1989.