MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 9, N° 1, 1980, pp. 125–127

ON SOME RESULTS OF C. A. MICCHELLI

by

I. RAŞA

(Cluj-Napoca)

1. We first recall some notations and results from [1] and [2]. Let X be a compact metrizable Hausdorff space. We will denote by $M^+(X)$ the cone of positive Radon measures on X. Let S be a closed linear subspace of $C(X)$ which contains a positive function. $U(S)$ will be the set of all elements of $M^+(X)$ which are uniquely determined by S; thus $\mu \in U(S)$ provided that whenever $\nu \in M^+(X)$ with $\mu(s) = \nu(s)$ for all $s \in S$ then $\nu = \mu$. The symbol ε_x will denote the Dirac measure defined by x. S will be called a Korovkin subspace of $C(X)$ if whenever (T_n) is a sequence of positive linear operators on $C(X)$ such that $\lim T_n s = s$ for all $s \in S$, it follows that $\lim T_n f = f$ for all $f \in C(X)$.

Let E be a locally convex linear topological space and K a compact convex metrizable subset of E. $A(K)$ will be the subspace of $C(K)$ consisting of all continuous affine functions on K. If Φ is a continuous convex function on K then Φ has a right Gateaux derivative, given by

$$D\Phi(x\,;\,y) = \lim_{t \downarrow 0} \frac{\Phi(x + ty) - \Phi(x)}{t}$$

for all x, y such that $x \in K$, $x + y \in K$.
We will say that Φ is smooth provided that for all $x \in K$ the mapping $y \to D\Phi(x\,;\,y - x)$ is in $A(K)$.

If K is a simplex we will denote by Π_x the unique boundary probability measure on K representing x.

If $\emptyset$ is a strictly convex continuous function on K, $S[\emptyset]$ will be the subspace of $C(K)$ generated by $\emptyset$ and $A(K)$.

Micchelli has proved:

PROPOSITION 1. *If $\emptyset$ is a strictly convex smooth function then* $\{\lambda\varepsilon_x | \lambda \in \mathbf{R}^+,\ x \in K\} \subset U(S[\emptyset])$.

THEOREM 1. *If $\emptyset$ is a strictly convex smooth function then $S[\emptyset]$ is a Korovkin subspace of $C(K)$.*

THEOREM 2. *If $\emptyset$ is a strictly convex smooth function on a Bauer simplex K, then*

$$U(S[\emptyset]) = \{\lambda\varepsilon_x | \lambda \in \mathbf{R}^+,\ x \in K\} \cup \{\lambda\Pi_x | \lambda \in \mathbf{R}^+,\ x \in K\}.$$

In fact, any smooth convex function $\emptyset$ has the property that $\emptyset(x) = \max\{a(x) \mid a \leq \emptyset,\ a \in A(K)\}$ for all $x \in K$; this is the property employed in the proofs of the above results.

2. The purpose of this note is to show that in Proposition 1, Theorem 1 and Theorem 2 the hypothesis that $\emptyset$ is smooth can be omitted.

We need the following

LEMMA *Let K be a compact convex metrizable subset of E, μ a probability measure on K with the barycenter $x \in K$, and $f \in C(K)$ a strictly convex function. If $\mu(f) = f(x)$ then $\mu = \varepsilon_x$.*

Proof. Let $\mu \neq \varepsilon_x$. Then there exists a $y \neq x$, $y \in \operatorname{supp}\mu$. Let Y be a closed convex neighbourhood of y, $x \notin Y$. Then $\mu(Y) < 0$. If $\mu(Y) = 1$ then the barycenter of μ will be in Y, but $x \notin Y$; thus $\mu(Y) < 1$. We will denote $\mu(Y)$ by a, $0 < a < 1$. Consider now the Radon measures defined by

$$\mu_1(B) = \frac{1}{a}\mu(B \cap Y),\quad \mu_2(B) = \frac{1}{1-a}\mu(B \cap (K \setminus Y))$$

for all Borel sets $B \subset K$. Then $\mu = \mu_1 a + (1-a)\mu_2$. Let x_i be the barycenter of μ_i, $i = 1, 2$. We have $\mu_1(Y) = 1$, hence $x_1 \in Y$ and $x_1 \neq x$. Clearly $x = ax_1 + (1-a)x_2$; if $x_1 = x_2$ then $x = x_1$, a contradiction. Thus $x_1 \neq x_2$. Now

$$f(x) = \mu(f) = a\mu_1(f) + (1-a)\mu_2(f) \geq af(x_1) + (1-a)f(x_2) >$$
$$> f(ax_1 + (1-a)x_2) = f(x).$$

This is a contradiction and the proof is complete.

Now we can obtain the following improved form of Proposition 1.

PROPOSITION 2. *If $\emptyset \in C(K)$ is a strictly convex function then*

$$\{\lambda\varepsilon_x | \lambda \in \mathbf{R}^+,\ x \in K\} \subset U(S[\emptyset]).$$

Proof. Let $x \in K$. It suffices to prove that $\varepsilon_x \in U(S[\emptyset])$. Let $\nu \in M^+(K)$, $\nu(s) = \varepsilon_x(s)$ for all $s \in S[\emptyset]$. Then $\nu(h) = h(x)$ for all $h \in A(K)$. It follows that x is the barycenter of ν. From $\nu(\emptyset) = \emptyset(x)$ it follows, using the above lemma, that $\nu = \varepsilon_x$. This completes the proof.

Proposition 2 and the characterization of the Korovkin subspaces already mentioned enable us to obtain a generalized version of Theorem 1:

THEOREM 3. *If $\emptyset \in C(K)$ is a strictly convex function then $S[\emptyset]$ is a Korovkin subspace of $C(K)$.*

Finally, using the results of Micchelli and Proposition 2, it is easy to prove that Theorem 2 holds without the hypothesis that $\emptyset$ is smooth.

REFERENCES

[1] Micchelli, C. A., *Convergence of positive linear operators on $C(X)$*. J. Approximation Theory, **13**, 305–315 (1975).

[2] Phelps, R. R., *Lectures on Choquet's theorem*. Math. Studies, Princeton, Van Nostrand, 1966.

Received 6. XII. 1979