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NON CONVEX OPTIMIZATION PROBLEMS ON WEAKLY COMPACT SUBSETS OF BANACH SPACES

by

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1. Introduction

Let X be a normed space, M a nonvoid subset of X and x an element of X.

The problem of nearest points. Let $d(x, M) = \inf\{||x - y|| : y \in M\}$ the distance from x to M, and let $P_M(x) = \{y \in M : ||x - y|| = d(x, M)\}$, the set (possibly empty) of nearest points to x in M (or the set of elements of best approximation of x by elements of M). Put $E(M) = \{x \in X : P_M(x) \neq \emptyset\}$ and $Tc(M) = \{x \in X : \text{card}(P_M(x)) = 1\}$. STEČKIN [15] proved that if X is a uniformly convex Banach space and M is a nonvoid closed subset of X, tehn $X \setminus Tc(M)$ is of first Baire category (in particular, $Tc(M)$ is dense in X). STEČKIN [15] asked if this result remains true in a locally uniformly convex Banach space. In [7] it was shown that the answer is no: there exists an equivalent locally uniformly convex norm p on c_0 (namely, Day's norm, see [8]) such that (c_0, p) contains a closed bounded symmetric convex body, such that $E(M) = M$ (such sets are called *antiproximinal*). Another solution (fortunately, also negative) was kindly comunicated to the author by Professor P. Kenderov: if X is a separable non reflexive Banach space, then, by a result of TROYANSKI [16] there exists on X an equivalent locally uniformly convex norm p. (X, p) being nonreflexive, by James' theorem there exists a continuous linear functional x' on X which does not attain its norm on the unit ball of (X, p). The corresponding closed hyperplane $H = (x')^{-1}(0)$ is antiproximinal in (X, p), i.e. $E(H) = H$. Recently KA-SING LAU [13] proved that Stečkin's result holds in reflexive locally uniformly convex Banach spaces.

The problem of farthest points. Suppose further the set M bounded and let $h(x, M) = \sup\{||x - y|| : y \in M\}$. Put $Q_M(x) = \{y \in M : ||x - y|| =$

$= h(x, M)\}$ — the set (possibly empty) of farthest points to x in M and $e(M) = \{x \in X : Q_M(x) \neq \emptyset\}$. EDELSTEIN [9] proved that if M is a nonvoid closed bounded subset of uniformly convex Banach space X, then $e(M)$ is dense in X. ASPLUND [1] extended this result proving that if M is a nonvoid closed bounded subset of a reflexive locally uniformly convex Banach space X, then $e(M)$ contains a G_δ set dense in X. Finally KA-SING LAU [12] proved a similar result for weakly compact subsets of arbitrary Banach spaces, and derived from this one, Asplund's theorem.

Perturbed problems. Let J be a real functional defined on M. BARANGER [2] considered the following extensions of the problems of nearest points and of farthest points: Problem J-inf (Problem J-sup): for $x \in X$ find $y_0 \in M$ such that $||x - y_0|| + J(y_0) = \inf\{||x - y|| + J(y) : y \in M\}$ ($= \sup\{||x - y|| + J(y) : y \in M\}$ respectively), and proved that if X is a uniformly convex Banach space, M a closed nonvoid subset of X and $J : M \to R$ is lower semicontinuous and bounded from below, then the set of all $x \in X$ for which the problem J-inf has a solution is dense in X. If X is a reflexive locally uniformly convex Banach space, M a nonvoid closed bounded subset of X and $J : M \to R$ is upper semicontinuous and bounded from above, then the set of all $x \in X$ for which the problem J-sup has a solution, contains a G_δ set dense in X. Other results along this line were obtained by BARANGER–TEMAM [3], BIDAUT [5], EKELAND–LEBOURG [11].

The aim of this Note is to extend Ka-Sing Lau's result on farthest points of weakly compact sets to perturbed problems (Problem J-sup, with an apropriate J). In the third section some applications to optimal control problems of systems governed by partial differential equations, are given.

2. The main result.

In this section we prove the following theorem:

2.1 THEOREM. *If X is a Banach space, M a nonvoid weakly compact subset of X and $J : M \to R$ is an upper semicontinuous and bounded from above functional, then the set of all $x \in X$ for which the problem J-sup has a solution contains a G_δ set dense in X.*

In this theorem, and in what follows, by „weak" we mean $\sigma(X, X')$, X' the dual of X. Our proof follows closely KA-SING LAU'S proof in [12].

Recall that if $f : X \to R \cup \{\infty\}$ is a function on X, a *subgradient* of f at a point $x \in X$ (such that $f(x) < \infty$) is a continuous linear functional x' such that

(2.1) $$x'(y - x) \leq f(y) - f(x),$$

for all $y \in X$. The set (possibly empty) of all subgradients of f at x is denoted by $\partial f(x)$ and is called the *subdifferential* of f at x. If f is continuous, at x then, $\partial f(x)$ is a nonempty weakly compact subset of X' (see [4]).

For $x \in X$ put

(2.2) $$r(x) = \sup\{||x - y|| + J(y) : y \in M\}.$$

2.2 LEMMA. *Let X a normed space, M a nonvoid bounded subset of X, $J : M \to R$ a bounded from above functional and let $r : X \to R$ be defined by (2.2). Then*

(i) *r is convex and Lipschitz, with constant 1, i.e.*

$$|r(x) - r(y)| \leq ||x - y||, \text{ for all } x, y \in X;$$

(ii) *if $x' \in \partial r(x)$ then $||x'|| \leq 1$, for all $x \in X$.*

Proof. (i). The functions $r_y : X \to R$, defined by $r_y(x) = ||x - y|| + J(y)$, are convex for all $y \in M$, and so will be their supremum r. Now, for $x, y \in X$ and $z \in M$

$$||x - z|| + J(z) \leq ||x - y|| + ||y - z|| + J(z) \leq ||x - y|| + r(y),$$

so that $r(x) \leq ||x - y|| + r(y)$, or $r(x) - r(y) \leq ||x - y||$. Interchanging the roles of x and y one obtains $|r(x) - r(y)| \leq ||x - y||$.

(ii). If $x' \in \partial r(x)$, then $x'(y - x) \leq r(y) - r(x) \leq ||y - x||$, for all $y \in X$, which implies $||x'|| \leq 1$. Lemma 2.2 is proved.

If $x' \in \partial r(x)$. then, by Lemma 2.2 (ii)

$$x'(y - x) - J(y) \geq - ||x - y|| - J(y) \geq - r(x),\ y \in M,$$

so that

(2.3) $$\inf\{x'(y - x) - J(y) : y \in M\} \geq - r(x),$$

for all $x \in X$. The following lemma shows that the equality sign holds in (2.3) for all $x \in X$, excepting a set of first Baire category.

2.3 LEMMA. *Let X be a Banach space, M a nonvoid closed bounded subset of X and let $J : M \to R$ be bounded from above. If $r(x)$ is defined by (2.2), then the set:*

$F = \{x \in X : \exists x' \in \partial r(x)$ such that $\inf\{x'(y - x) - J(y) : y \in M\} > - r(x)\}$ is of F_δ type and of first Baire category.

Proof. For $n \in N$, let $F_n = \left\{x \in X : \exists\, x' \in \partial r(x) \text{ such that } \inf\{x'(y - x) - J(y) : y \in M\} \geq - r(x) + \frac{1}{n}\right\}$. Obviously $F = \bigcup_{n=1}^{\infty} F_n$. Therefore, to prove Lemma 2.3, it is sufficient to show that

(a) F_n is closed in X; and

(b) $\operatorname{int} F_n = \emptyset$,

for all $n \in N$.

(a). Let $\{x_k : k \in N\}$ be a sequence in F_n converging to a point $x \in X$. For each $k \in N$, choose $x'_k \in \partial r(x_k)$ such that

(2.4) $$\inf\{x'_k(y - x_k) - J(y) : y \in M\} \geq - r(x_k) + \frac{1}{n}.$$

Since $\|x'_k\| \leq 1$, $k \in N$, (Lemma 2.2 (ii)), the sequence $\{x'_k,\ k \in N\}$ admits a subnet $\{x'_i,\ i \in I\}$ $\sigma((X', X)$ — convergent to an element x' of X', with $\|x'\| \leq 1$. For $z \in X$, we have

$$|x'_i(z - x_i) - x'(z - x)| \leq |x'_i(z - x_i) - x'_i(z - x)| + |x'_i(z - x)| -$$
$$- x'(z - x)| \leq \|x_i - x\| + |(x'_i - x')(z - x)|$$

for all $i \in I$, which shows that the net $\{x'_i(z - x_i)\}$ converges to $x'(z - x)$. Since $x'_i \in \partial r(x_i)$, we have $x'(z'_i - x_i) + r(x_i) \leq r(z)$ (see 2.1)) so that $x'(z - x) + r(x) \leq r(z)$, for all $z \in X$, which shows that $x' \in \partial r(x)$. From $x'_i(y - x) - J(y) \geq - r(x) + \frac{1}{n}$, follows $x'(y - x) - J(y) \geq - r(x) + \frac{1}{n}$ for all $y \in M$. Therefore $x \in F$ and the set F_n is closed.

(b) $\operatorname{int} F_n = \emptyset$, $n = 1, 2, \ldots$

Suppose that there exist $k \in N$, $y_0 \in F_n$ and a ball U of center y_0 included in F_k. The set M being bounded there exists $\lambda > 0$ such that

$$x = y_0 + \lambda(y_0 - z) \in U, \tag{2.4}$$

for all $z \in M$. Let $\varepsilon = \lambda[(\lambda + 1)k]^{-1}$ and let $z_0 \in M$ be such that

$$r(y_0) - \varepsilon \leq \|y_0 - z_0\| + J(z_0) \leq r(y_0) \tag{2.5}$$

and let

$$x_0 = y_0 + \lambda(y_0 - z_0). \tag{2.6}$$

Since by (2.4), $x_0 \in U \subset F_k$, it follows the existence of a $x' \in \partial r(x_0)$ such that

$$\inf\{x_0(z - x_0) - J(z_0) : z \in M\} \geq - r(x_0) + \frac{1}{k}. \tag{2.7}$$

By (2.5), $r(y_0) - r(x_0) \leq \|y_0 - z_0\| + J(z_0) + \varepsilon - r(x_0)$, and by (2.6), $y_0 - z_0 = (\lambda + 1)^{-1}(x_0 - z_0)$. Therefore

$$r(y_0) - r(x_0) \leq (\lambda + 1)^{-1}\|x_0 - z_0\| + J(z_0) + \varepsilon - r(x_0) \leq$$
$$\leq \lambda(\lambda + 1)^{-1}[r(x_0) - J(z_0)] + J(z_0) + \varepsilon - r(x_0) =$$
$$= -\lambda(\lambda + 1)^{-1}r(x_0) + \lambda(\lambda + 1)^{-1}J(z_0) + \varepsilon =$$
$$= \lambda(\lambda + 1)^{-1}[-r(x_0) + J(z_0)] - \varepsilon \leq$$
$$\leq \lambda(\lambda + 1)^{-1}x'(z_0 - x_0) - \lambda[(\lambda + 1)k]^{-1} + \varepsilon.$$

But

$$z_0 - x_0 = y_0 - x_0 + z_0 - y_0 = y_0 - x_0 + \lambda^{-1}(y_0 - x_0) = (\lambda + 1)\lambda^{-1}$$
$$(y_0 - x_0),$$

so that

$$r(y_0) - r(x_0) < x_0(y_0 - x_0) - \lambda[(\lambda + 1)k]^{-1} + \varepsilon = x'_0(y_0 - x_0),$$

in contradiction to $x_0 \in r(x_0)$.

Proof of Theorem 2.1. Let F be the set defined in Lemma 2.3 and let $D = X \setminus F$. Obviously, D is a G_δ set and by the Baire category theorem, D is dense in X. For $x \in D$ and $x' \in \partial r(x)$, we have

$$\inf\{x'(y - x) - J(y) : y \in M\} = - r(x). \tag{2.7}$$

Since J is weakly upper semicontinuous, the function $\rho(y) = x'(y - x) - J(y)$, $y \in M$, is weakly lower semicontinuous. Taking into account this fact and the weak compactity of M, it follows the existemce of a point $y_0 \in M$, such that $\rho(y_0) = \inf\{\rho(y) : y \in M\}$. But then, by (2.6)

$$- r(x) = x'(x - y_0) - J(y_0) \geq - \|x - y_0\| - J(y_0) \geq - r(x).$$

Therefore, $r(x) = \|x - y_0\| + J(y_0)$, and Theorem 2.1 is proved.

3. The optimal control problem

Let U be a Banach space (the control space), U_{ad} a weakly compact subset of U (the set of admissible controls) and H a Banach space (the space of observations). One suppose the state of the system given by

$$y = Gu + z,$$

where z is a fixed element in H and $G : U \to H$ is a continuous linear operator.

3.1. PROPOSITION. *For every* $\varepsilon > 0$ *the set of all* $x \in U$ *for which there exists* $u_0 \in U_{ad}$ *such that*

$$-\|y(u_0) - z\| + \varepsilon\|u_0 - x\| = \sup\ \{-\|y(u) - z\| + \varepsilon\|u - x\| : u \in U_{ad}\},$$

contains a G_δ *set dense in* U.

Proof. The operator $G : U \to H$, being linear and continuous, will be continuous also with respect to weak topologies $\sigma(U, U')$, $\sigma(H, H')$ on U and H, respectively. Since the norm on a normed space is a weakly lower semicontinuous functional, $J(u) = -\|y(u) - z\|$ will be weakly upper semicontinuous, and Theorem 2.1 can be applied to obtain the desired result.

Let now Ω be an open bounded subset of R with smooth boundary. Consider the differential operator

$$Ly = -\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial y}{\partial x_i}\right) + \Sigma_i \frac{\partial}{\partial x_i}(a_i y) + ay, \tag{3.1}$$

where $a_{ij},\ a_i \in C^1(\overline{\Omega})$, $a \in L_\infty(\Omega)$,

$$a \geq \beta > 0,\ a + \Sigma_i \frac{\partial a_i}{\partial x_i} \geq \beta > 0, \text{ a.e. in } \Omega \tag{3.2}$$

and

(3.3) $$\Sigma_i a_i n_i \geq 0 \text{ on } \Gamma,$$

where $n = (n_1, \ldots, n_m)$ is the unit outward normal on the boundary of Γ.

Denote, also

(3.4) $$\frac{\partial}{\partial n_L} = \Sigma_{i,j}\, a_{ij} n_j \frac{\partial}{\partial x_i}.$$

Let $y \in W^{1,1}(\Omega)$, for $f \in L^1(\Omega)$, $u \in L^1(\Gamma)$, be a *weak solution* of the Neumann problem:

(3.5) $$\begin{cases} Ly = f \text{ in } \Omega \\ \dfrac{\partial y}{\partial n_L} = u \text{ on } \Gamma \end{cases}$$

i.e.

(3.6) $$a(y, v) := \int_\Omega \Big[\Sigma_{i,j}\, a_{ij} \frac{\partial y}{\partial x_i}\frac{\partial v}{\partial x_j} + \Sigma_i \frac{\partial}{\partial x_i}(a_i y) + ayv\Big]\mathrm{d}x =$$

$$= \int_\Omega fv \,\mathrm{d}x + \int_\Gamma uv \,\mathrm{d}\sigma,$$

for all $v \in C^1(\overline{\Omega})$.

Suppose taht the following inequality holds

(3.7) $$\Sigma_{i,j}\, a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2 \text{ a.e. } x \in \Omega,$$

for all $\xi \in R^n$.

Consider the following optimal control problem : find $u_0 \in U_{ad}$ such that

(3.8) $$\sup\{-\|y(u) - z\|_{1,q} + \varepsilon\|u - v\|_{L^1(\Gamma)} : u \in U_{ad}\} =$$

$$= -\|y(u_0) - z\| + \varepsilon\|u_0 - v\|,$$

where U_{ad} is a weakly compact subset of $L^1(\Gamma)$, $1 \leq q \leq n/(n-1)$ is fixed, $y(u)$ is a weak solution of problem (3.5) and $v \in L^1(\Gamma)$.

By a result of BREZIS and STRAUSS [6] the problem (3.5) has a unique weak solution $y(u)$ for all $u \in L^1(\Gamma)$ and $y(u) \in W^{1,q}(\Omega)$, for $1 \leq q \leq n/(n-1)$.

Furthermore, the following inequality

(3.9) $$\|y\|_{1,q} \leq C_q(\|f\|_{L^1(\Omega)} + \|u\|_{L^1(\Gamma)})$$

holds (see Lemma 23 in [6]).

If (u_k) is a sequence in $L^1(\Gamma)$ converging to $u \in L^1(\Gamma)$, then $y(u) - y(u_k)$ is the unique weak solution of Neumann problem:

$$\begin{cases} Ly = 0 \text{ in } \Omega \\ \dfrac{\partial y}{\partial n_N} = u - u_k \text{ on } \Gamma. \end{cases}$$

By (3.9)

$$\|y(u) - y(u\ )\|_{1,q} \leq C_q\|u - u_k\|_{L^1(\Gamma)} \to 0, \text{ for } k \to \infty,$$

which shows that the application $u \to y(u)$ from $L^1(\Gamma)$ to W^1 (Ω) is continuous.

The application $u \to y(u)$ being affine, like in the proof of Proposition 3.1, follows the weak lower semicontinuity of the functional $J(u) = \|y(u) - z\|$. By a direct application of Theorem 3.1, the set of all $v \in L^1(\Gamma)$ for which the problem (3.8) has a solution contains a G_δ set dense in $L^1(\Gamma)$.

REFERENCES

[1] Asplund, E., *Farthest points in reflexive locally uniformly rotund Banach spaces*, Israel J. Math. **4**, 213–216 (1966).

[2] Baranger, J., *Existence de solutions pour des problèmes d'optimisation non convexe*, J. Math. Pures et Appl. **52**, 377–405 (1973).

[3] Baranger, J., Teman, R., *Non convex optimization problems depending on a parameter*, SIAM J. Control **13**, 146–152 (1975).

[4] Barbu, V., Precupanu, Th., *Convexity and optimization in Banach spaces*, Ed. Academiei and Sijthoff Noordhoff, 1978.

[5] Bidaut, M.F., *Existence theorems for usual and approximate solutions of optimal control problems*, JOTA **15**, 393–412 (1975).

[6] Brezis, H., Strauss, W. A., *Semi-linear second-order elliptic equations in L^1*, J. Math. Soc. Japan, **25**, 565–590, (1973).

[7] Cobzaş, S., *Antiproximinal sets in some Banach spaces*, Math. Balkanica, **4**, 79–82 (1974).

[8] Day, M. M., *Strict convexity and smoothness of normed spaces*, Trans. Amer. Math. Soc. **78**, 516–528 (1955).

[9] Edelstein, M., *Farthest points of sets in uniformly convex Banach spaces*, Israel J. Math. **4**, 171–176 (1966).

[10] Edelstein, M., *On nearest points in uniformly convex Banach spaces*, J. London Math. Soc. **43**, 375–377, (1968).

[11] Ekeland, I., Lebourg, G., *Generic Fréchet-differentiability and perturbed optimization problems in Banach spaces*, Trans. Amer. Math. Soc. **224**, 193–216 (1976).

[12] Lau, Ka-Sing, *Farthest points in weakly compact sets*, Israel J. Math. **22**, 168–174, (1975).

[13] Lau, Ka-Sing, *Almost Chebyshev subsets in reflexive Banach spaces*, Indiana Univ. Math. J. **27**, 791–795 (1978).

[14] Lions, J. L., *Contrôle optimal de systèmes gouvernés par des équations aux derivée partielles*, Gauthier-Villars, Paris 1998.

[15] Стечкин С. Б., *Аппроксимативные свойства множеств в линейных нормированных пространствах*, Rev. Math. Pures et Appl. *(Bucharest)*, **8**, 5–18 (1963)

[16] Troyanski, S., *On locally uniformly convex and differentiable norms in certain non separable Banach spaces*, Studia Math. **37**, 173–180.

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