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A BOOLEAN METHOD IN BIVARIATE INTERPOLATION

by

F.J. DELVOS and H. POSDORF

(Siegen, West Germany) (Bochum, West Germany)

0. Introduction

Boolean methods of multivariate interpolation were introduced in the fundamental papers of GORDON [4, 5]. It was shown in these papers that any collection of commutative projectors (in particular interpolation projectors) generates a distributive lattice each of whose elements provides a method for approximating a multivariate function. In particular, every (finite) lattice has a unique maximal element (i.e. projector) which is characterized by having the largest range. The present paper is concerned with the analysis of a ,,locally'' maximal projector corresponding to a finite family of projectors of a certain bivariate approximation lattice whose generatorrs are parametric extensions of univariate interpolation projectors. The interpolation method under consideration is called *generalized Biermann interpolation* since the classical Biermann formula for interpolating a function on a triangular mesh turns out to be a special case (STANCU [8]). Using certain properties of the generating projectors of generalized Biermann interpolation we are able to derive an explicit expression for the cardinal functions. After showing that the bivariate Taylor formula and the Biermann formula have the same Boolean structure we apply generalized Biermann interpolation to reduced Hermite interpolation introduced by MELKES [6] in the finite element method. In particular we derive an explicit expression for the cardinal functions of reduced Hermite interpolation. Finally we prove a remainder formula for generalized Biermann interpolation which is applied to reduced Hermite interpolation.

1. The Biermann projector

Let $S = [a, b] \times [a, b]$ be a square and let $B(S)$ denote a subspace of the space $C(S)$ of continuous functions on S. Furthermore, let

$$\{A'_i \quad 1 \le i \le N\}$$

be a set of N partial linear operators which operate on $f \in B(S)$ as a function of x such that

$$A'_i(f) = a'_i(y) \in B(S).$$

Similarly, let

$$\{A''_i : 1 \le i \le N\}$$

be a set of N partial linear operators which operate on $f \in B(S)$ as a function of y such that

$$A''_i(f) = a''_i(x) \in B(S).$$

Now consider the finite sequence of natural numbers

$$1 \le n_1 \le n_2 \le \ldots \le n_{K-1} \le n_K = N$$

and let

$$\Lambda'_k = \{L'_{j,n_k} : 1 \le j \le n_k\}, \quad 1 \le k \le K$$

be a set of univariate cardinal functions, with respect to

$$\mathsf{A}'_k = \{A'_i : 1 \le i \le n_k\}, \quad 1 \le k \le K,$$

i.e.

(1) $$A'_i(L'_{j,n_k}) = \delta_{i,j}, \quad 1 \le i, \; j \le n_k, \quad 1 \le k \le K.$$

Furthermore, we suppose that

(2) $$\langle \Lambda'_k \rangle \subset \langle \Lambda'_{k+1} \rangle, \quad 1 \le k \le K-1.$$

Similarly let

$$\Lambda''_k = \{L''_{j,n_k} : 1 \le j \le n_k\}, \quad 1 \le k \le K$$

be a set of univariate cardinal functions with respect to

$$\mathsf{A}''_k = \{A''_i : 1 \le i \le n_k\}, \quad 1 \le k \le K,$$

i.e.

(3) $$A''_i(L''_{j,n_k}) = \delta_{i,j}, \quad 1 \le i, \; j \le n_k, \quad 1 \le k \le K,$$

and suppose that

(4) $$\langle \Lambda''_k \rangle \subset \langle \Lambda''_{k+1} \rangle, \quad 1 \le k \le K-1.$$

Thus we can construct the set

$$\{P'_{n_k} : 1 \le k \le K\}$$

of parametric projectors P'_{n_k}, $1 \le k \le K$, associated with the sets Λ'_k and A'_k, $1 \le k \le K$:

(5) $$P'_{n_k}(f) = \sum_{i=1}^{n_k} A'_i(f)\, L'_{i,n_k}(x), \quad 1 \le k \le K.$$

Similarly we construct the set

$$\{P_{n_k} : 1 \le k \le K\}$$

of parametric projectors P''_{n_k}, $1 \le k \le K$, associated with the sets Λ''_k and A''_k, $1 \le k \le K$:

(6) $$P''_{n_k}(f) = \sum_{i=1}^{n_k} A''_i(f)\, L''_{i,n_k}(y), \quad 1 \le k \le K.$$

It follows from the relations (1), ..., (4) that the projectors P'_{n_k} and P''_{n_k}, $1 \le k \le K$, are absorptive, i.e.

(7) $$P'_{n_i}P'_{n_j} = P'_{n_j}P'_{n_i} = P'_{n_i}, \quad 1 \le i \le j \le K,$$
$$P''_{n_i}P''_{n_j} = P''_{n_j}P''_{n_i} = P''_{n_i}, \quad 1 \le i \le j \le K.$$

Finally we assume that the sets A'_K, A''_K commute on $B(S)$:

(8) $$A'_i A''_j(f) = A''_j A'_i(f), \quad 1 \le i, \; j \le n_K, \quad f \in B(S).$$

Thus it follows from (2), (4), (7), (7) that the projectors $P'_{n_1}, \ldots, P'_{n_K}$, $P''_{n_1}, \ldots, P''_{n_K}$ commute and generate a distributive lattice (cf. GORDON [4]). Special elements of this lattices are product projectors $P'_{n_i}P''_{n_j}$, $1 \le i, \; j \le K$:

$$P'_{n_i}P''_{n_j}(f) = \sum_{l=1}^{n_i} \sum_{m=1}^{n_j} A'_l A''_m(f)\, L'_{l,n_i}(x)\, L''_{m,n_j}(y).$$

Obviously, the product projectors enjoy the following interpolation properties:

(9) $$A'_l A''_m(P'_{n_i}P''_{n_j}(f)) = A'_l A''_m(f)$$
$$(1 \le l \le n_i, \; 1 \le m \le n_j, \quad 1 \le i, \; j \le K).$$

DEFINITION 1. *The generalized Biermann projector* P_K *is defined as the Boolean sum of the projectors* $P'_{n_r} P''_{n_{K+1-r}}$, $1 \le r \le K$:

$$P_K = P'_{n_1} P''_{n_K} \oplus P'_{n_2} P''_{n_{K-1}} \oplus \dots \oplus P'_{n_{K-1}} P''_{n_2} \oplus P'_{n_K} P''_{n_1}.$$

(Note that the Boolean sum of two commuting projectors A and B is defined by $A \oplus B = A + B - AB$).

The generalized Biermann projector P_K *is locally maximal, since it is the maximal element of the sublattice generated by* $P'_{n_r} P''_{n_{K+1-r}}$, $1 \le r \le K$.

It enjoys the interpolation properties given in

THEOREM 1. *For* $f \in B(S)$, *the function* $g = P_K(f)$ *interpolates* f *in the sense that*

$$A'_i A''_j (g) = A'_i A''_j (f), \quad (1 \le i \le n_r, \ 1 \le j \le n_{K+1-r}, \ 1 \le r \le K).$$

Proof. The lattice theoretical construction of P_K yields

$$P'_{n_r} P''_{n_{K+1-r}} P_K = P'_{n_r} P''_{n_{K+1-r}}, \quad 1 \le r \le K.$$

Thus the interpolation properties of the Boolean sum projector P_K follow immediately from the corresponding properties of the product projectors (see (9)).

Our next objective is to derive an explicit expression of the projector P_K in terms of sums of product projectors $P'_{n_r} P''_{n_{K+1-r}}$, $1 \le r \le K$.

LEMMA 1. *For the generalized Biermann projector* P_K *the following representation formula is valid:*

$$P_K = \sum_{r=0}^{K-1} P'_{n_{r+1}} P''_{n_{K-r}} - \sum_{r=1}^{K-1} P'_{n_r} P''_{n_{K-r}} \tag{10}$$

Proof. We will prove this representation formula for $K = 3$. The general case can be proved by induction (cf. DELVOS–POSDORF [3]).

$$P_3 = P'_{n_1} P''_{n_3} \oplus P'_{n_2} P''_{n_2} \oplus P'_{n_3} P''_{n_1} =$$
$$= (P'_{n_1} P''_{n_3} + P'_{n_2} P''_{n_3} - P'_{n_1} P''_{n_2}) \oplus P'_{n_3} P''_{n_1} =$$
$$= P'_{n_1} P''_{n_3} + P'_{n_2} P''_{n_2} - P'_{n_1} P''_{n_2} + P'_{n_3} P''_{n_1} - P'_{n_1} P''_{n_1} - P'_{n_2} P''_{n_1} + P'_{n_1} P''_{n_1} =$$
$$= P'_{n_1} P''_{n_3} + P'_{n_2} P''_{n_2} + P'_{\ 3} P''_{n_1} - P'_{n_1} P''_{n_2} - P'_{n_2} P''_{n_1}$$

THEOREM 2. *Let* $f \in B(S)$, *then the generalized Biermann interpolatnt* $P_K(f)$ of f *has the following representation* $(n_0 = 0)$:

$$P_K(f)(x, y) = \sum_{s=0}^{K-s} \sum_{r=0}^{s} \sum_{i=n_r+1}^{n_{r+1}} \sum_{j=n_{K-1-s}+1}^{n_{K-s}} A'_i A''_j (f) L_{i,j}(x, y), \tag{11}$$

where $L_{i,j}(x, y)$ *denote the cardinal functions with respect of* $A'_i A''_j$; $(n_r + 1 \le i \le n_{r+1}, \ n_{K-1-s} + 1 \le j \le n_{K-s}, \ 0 \le r \le s \le K-1)$ *given by*

$$L_{i,j}(x, y) = \sum_{l=r}^{s} L'_{i,n_{l+1}}(x) L''_{j,n_{K-l}}(y) - \sum_{l=r+1}^{s} L'_{i,n_l}(x) L''_{j,n_{K-l}}(y). \tag{12}$$

Proof. Taking into account (5) and (6) the representation formula (10) yields

$$P_K(f)(x,y) = \sum_{r=0}^{K-1} \sum_{i=1}^{n_{r+1}} \sum_{j=1}^{n_{K-r}} A'_i A''_j (f) L'_{i,n_{r+1}}(x) L''_{j,n_{K-r}}(y) -$$
$$- \sum_{r=1}^{K-1} \sum_{i=1}^{n_r} \sum_{j=1}^{n_{K-r}} A'_i A''_j (f) L'_{i,n_r}(x) L''_{j,n_{K-r}}(y).$$

Thus we obtain

$$P_K(f)(x,y) = \sum_{r=0}^{K-1} \sum_{i=n_r+1}^{n_{r+1}} \sum_{s=r}^{K-1} \sum_{j=n_{K-1-s}+1}^{n_{K-s}} A'_i A''_j (f) \left(\sum_{l=r}^{s} L'_{i,n_{l+1}}(x) L''_{j,n_{K-l}}(y) \right) -$$
$$- \sum_{r=1}^{K-1} \sum_{i=n_{r-1}+1}^{n_r} \sum_{s=r}^{K-1} \sum_{j=n_{K-1-s}+1}^{n_{K-s}} A'_i A''_j (f) \left(\sum_{l=r}^{s} L'_{i,n_l}(x) L''_{j,n_{K-l}}(y) \right) =$$
$$= \sum_{r=0}^{K-1} \sum_{i=n_r+1}^{n_{r+1}} \sum_{s=r}^{K-1} \sum_{j=n_{K-1-s}+1}^{n_{K-s}} A'_i A''_j (f) \left(\sum_{l=r}^{s} L'_{i,n_{l+1}}(x) L''_{j,n_{K-l}}(y) \right) -$$
$$- \sum_{r=0}^{K-2} \sum_{i=n_r+1}^{n_{r+1}} \sum_{s=r+1}^{K-1} \sum_{j=n_{K-1-s}+1}^{n_{K-s}} A'_i A''_j (f) \left(\sum_{l=r+1}^{s} L'_{i,n_l}(x) L''_{j,n_{K-l}}(y) \right) =$$
$$= \sum_{r=0}^{K-1} \sum_{i=n_r+1}^{n_{r+1}} \sum_{s=r}^{K-1} \sum_{j=n_{K-1-s}+1}^{n_{K-s}} A'_i A''_j (f) \left\{ \sum_{l=r}^{s} L'_{i,n_{l+1}}(x) L''_{j,n_{K-l}}(y) - \right.$$
$$\left. - \sum_{l=r+1}^{s} L'_{i,n_l}(x) L''_{j,n_{K-l}}(y) \right\} =$$
$$= \sum_{s=0}^{K-1} \sum_{r=0}^{s} \sum_{i=n_r+1}^{n_{r+1}} \sum_{j=n_{K-1-s}+1}^{n_{K-s}} A'_i A''_j (f) \left\{ \sum_{l=r}^{s} L'_{i,n_{l+1}}(x) L''_{j,n_{K-1}}(y) - \right.$$
$$\left. - \sum_{l=r+1}^{s} L'_{i,n_l}(x) L''_{j,n_{K-l}}(y) \right\}.$$

Thus the equalities (11) and (12) are verified. We now merely have to prove that the $L_{i,j}(x, y)$ given in (12) are the desired cardinal functions.

Theorem 1 yields for $f \in B(S)$

$$A_i' A_j''(f) = A_i' A_j'' (P_K(f)); \quad (1 \le i \le n_r, 1 \le j \le n_{K+1-r}, 1 \le r \le K).$$

Let

$$\tilde{f}(x, y) := L'_{k,n_K}(x)\, L''_{l,n_K}(y), \quad 1 \le k, l \le n_K.$$

Then

$$A_i' A_j'' (\tilde{f}) = \delta_{i,k}\, \delta_{j,l}, \ (1 \le k \le n_r, \ 1 \le l \le n_{K+1-r}, \ 1 \le r \le K).$$

Thus

$$P_K(\tilde{f}) = L_{k,l}$$

which implies

$$A_i' A_j'' (L_{k,l}) = A_i' A_j'' (P_K(\tilde{f})) = A_i' A_j'' (\tilde{f}) = \delta_{i,k}\, \delta_{j,l}.$$

This proves our theorem.

We consider now in greater detail the two cases $n_r = r$ and $n_r = 2r$.

I: $n_r = r$. In this case we have $N = K$. An application of Theorem 2 yields:

$$P_N(f)(x, y) = \sum_{s=0}^{N-1} \sum_{r=0}^{s} A'_{r+1} A''_{N-s}(f) L_{r+1,N-s}(x, y),$$

with

$$L_{r+1,N-s}(x, y) = \sum_{i=r}^{s} L'_{r+1,\, i+1}(x) L''_{N-s,N-i}(y) - \sum_{i=r+1}^{s} L'_{r+1,i}(x) L''_{N-s,N-i}(y).$$

Note that the well known bivariate Taylor formula and the Biermann formula (STANCU [8]) are obtained by specializing the operators and cardinal functions as follows:

(i) Taylor formula:

$$A_i'(f) = D_x^{i-1} f(x_0, y), \ A_i''(f) = D_y^{i-1} f(x, y_0), \ 1 \le i \le N.$$

$$L'_{i,k}(x) = (x - x_0)^{i-1}/(i-1)! \qquad L''_{i,k}(y) = (y - y_0)^{i-1}/(i-1)! \qquad (1 \le i \le k, \ 1 \le k \le N).$$

(ii) Biermann formula:

$$A_i'(f) = f(x_i, y), \ A_i''(f) = f(x, y_i), \quad 1 \le i \le N.$$

$$L'_{i,k}(x) = \prod_{\substack{j=1\\ j \ne i}}^{k} (x - x_j)/(x_i - x_j) \qquad L''_{i,k}(y) = \prod_{\substack{j=1\\ j \ne i}}^{k} (y - y_j)/(y_i - y_j) \qquad (1 \le i \le k, \ 1 \le k \le N).$$

with $x_1 < x_2 < \ldots < x_N$, $y_1 < y_2 < \ldots < y_N$.

II: $n_r = 2r$. In this case we have $N = 2K$. An application of Theorem 2 yields:

$$(13) \qquad P_K(f)(x,y) = \sum_{s=0}^{K-1} \sum_{r=0}^{s} \sum_{i=2r+1}^{2r+2} \sum_{j=2(K-s)-1}^{2K-2s} A_i' A_j''(f) L_{i,j}(x, y),$$

with

$$(14) \qquad L_{i,j}(x, y) = \sum_{l=r}^{s} L'_{i,2(l+1)}(x) L''_{j,2(K-l)}(y) - \sum_{l=r+1}^{s} L'_{i,2l}(x) L''_{j,2(K-l)}(y).$$

From (13) the following formula can easily be derived:

$$(15) \qquad P_K(f)(x, y) = \sum_{s=0}^{K-1} \sum_{r=0}^{s} \{A'_{2(r+1)} A''_{2(K-s)}(f) L_{2(r+1),\, 2(K-s)}(x, y) + A'_{2(r+1)} A''_{2(K-s)-1}(f) L_{2(r+1),\, 2(K-s)-1}(x, y) + A'_{2r+1} A''_{2(K-s)}(f) L_{2r+1,\, 2(K-s)}(x, y) + A'_{2r+1} A''_{2(K-s)-1}(f) L_{2r+1,2(K-s)-1}(x, y)\}.$$

We will now apply this formula to reduced Hermite interpolation schemes of type II proposed by MELKES [6]. For this purpose let ($S = [0, 1] \times [0, 1]$):

$$A'_{2i-1}(f) = D_x^{i-1} f(0, y), \ A'_{2i}(f) = D_x^{i-1} f(1, y) \qquad A''_{2i-1}(f) = D_y f(x, 0), \ A''_{2i}(f) = D_y^{i-1} f(x, 1) \qquad (1 \le i \le K).$$

The corresponding cardinal functions are

$$(16) \qquad L'_{2j-1,2k}(x) = \frac{x^{j-1}}{(j-1)!}(1-x)^k \left(\sum_{s=0}^{k-j} \binom{k-1-s}{s} x^s \right)$$

$$(17) \qquad L''_{2j,2k}(x) = (-1)^{j-1} L'_{2j-1,2k}(1-x). \qquad 1 \le j \le k, \ 1 \le k \le K)$$

The functions $L''_{2j-1,2k}(y)$ and $L''_{2j,2k}(y)$ are defined analogously. (For the derivation of the cardinal functions see PHILLIPS [7]).

From (14), ..., (17) we now obtain the following

THEOREM 3. *The representation formula for the reduced Hermite interpolant $P_K(f)$ of f is given by:*

$$P_K(f)(x,y) = \sum_{s=0}^{K-1} \sum_{r=0}^{s} \{D_x^r D_y^{K-1-s} f(1,1)(-1)^{K-1-s+r} L_{2r+1,\, 2(K-s)-1}(1-x, 1-y) + D_x^r D_y^{K-1-s} f(1, 0)(-1)\, L_{2r+1,2(K-s)-1}(1-x, y) + D_x^r D_y^{K-1-s} f(0, 1)(-1)^{K-1-s} L_{2r+1,2(K-s)-1}(x, 1-y) + D_x^r D_y^{K-1-s} f(0, 0)\, L_{2r+1,2(K-s)-1}(x, y)\},$$

with

$$L_{2r+1,2(K-s)-1}(x, y) = \sum_{l=r}^{s} L'_{2r+1,2(l+1)}(x)\, L''_{2(K-s)-1,2(K-l)}(y) -$$

$$- \sum_{l=r+1}^{s} L'_{2r+1,2l}(x)\, L''_{2(K-s)-1,2(K-1)}(y).$$

2. Remainder representation

In this section we will derive a remainder representation formula for generalized Biermann projector P_K and apply this formula to reduced Hermite interpolation.

From Lemma 1 we obtain the following representation formula of P_K:

$$P_K = \sum_{r=0}^{K} (P'_{n_{r+1}} - P'_{n_r})\, P''_{n_{K-r}} \tag{18}$$

with $P'_{n_0} = P''_{n_0} = 0$ and $P'_{n_{K+1}} = I$ where 0 denotes the zero-projector and I denotes the identity-projector.

This leads us to

LEMMA 2. *For the remainder projector of* P_K, *denoted by* $\overline{P}_K$, *the following representation formula is valid:*

$$\overline{P}_K = \sum_{r=0}^{K} \overline{P'_{n_r}}\, \overline{P''_{n_{K-r}}} - \sum_{r=0}^{K-1} \overline{P'_{n_{r+1}}}\, \overline{P''_{n_{K-r}}}.$$

Proof. Taking into account the formula (18) the lemma is proved by:

$$\sum_{r=0}^{K} (P'_{n_{r+1}} - P'_{n_r})\, P''_{n_{K-r}} + \sum_{r=0}^{K} \overline{P'_{n_r}}\, \overline{P''_{n_{K-r}}} - \sum_{r=0}^{K-1} \overline{P'_{n_{r+1}}}\, \overline{P''_{n_{K-r}}} =$$

$$= \sum_{r=0}^{K} (P'_{n_{r+1}} - P'_{n_r})\, P''_{n_{K-r}} + \sum_{r=0}^{K} (\overline{P'_{n_r}} - \overline{P'_{n_{r+1}}})\, \overline{P''_{n_{K-r}}} =$$

$$= \sum_{r=0}^{K} (P'_{n_{r+1}} - P'_{n_r})\, P''_{n_{K-r}} + \sum_{r=0}^{K} (P'_{n_{r+1}} - P'_{n_r})\, \overline{P''_{n_{K-r}}} =$$

$$= \sum_{r=0}^{K} (P'_{n_{r+1}} - P'_{n_r}) = I.$$

An application of Lemma 2 to reduced Hermite interpolation yields

THEOREM 4. *Let* $f \in B(S) = C^{2K+2}(S)$, $S = [0, 1] \times [0, 1]$. *Then the remainder* $\overline{P}_K(f)$ *of the reduced Hermite interpolant* $P_K(f)$ *is given by:*

$$\overline{P}_K(f)(x, y) = \frac{y^K(y-1)^K}{(2K)!} D_y^{2K} f(x, y_K) +$$

$$+ \sum_{r=1}^{K-1} \frac{x^r(x-1)^r y^{K-r}(y-1)^{K-r}}{(2r)!\,(2(K-r))!} D_x^{2r} D_y^{2(K-r)} f(x_r, y_{K-r}) + \frac{x^K(x-1)^K}{(2K)!} D_x^{2K} f(x_K, y) -$$

$$- \sum_{r=0}^{K-1} \frac{x^{r+1}(x-1)^{r+1} y^{K-r}(y-1)^{K-r}}{(2(r+1))!\,(2(K-r))!} D_x^{2(r+1)} D_y^{2(K-r)} f(\tilde{x}_{r+1}, y_{K-r}).$$

Proof. Using the error estimates of univariate Hermite interpolation (cf. DAVIS [2]) the theorem is proved by

$$\overline{P}_K = \overline{P''_{2K}} + \sum_{r=1}^{K-1} \overline{P'_{2r}}\, \overline{P''_{2(K-r)}} + \overline{P'_{2K}} - \sum_{r=0}^{K-1} \overline{P'_{2(r+1)}}\, \overline{P''_{2(K-r)}},$$

which is an immediate consequence of Lemma 2.

3. Concluding remarks

In a recent paper WATKINS and LANCASTER [9] extended the method of reduced Hermite interpolation to fill up certain gaps in the finite element construction of rectangular elements. It is the topic of a forthcoming paper to show that the elements of this extended Melkes family can be constructed using generalized Biermann interpolation. In particular this will include the explicit construction of the corresponding cardinal functions.

APPENDIX

As an instance we present the interpolation stencils of reduced Hermite interpolation for $K = 2, 3, 4$. A dot surrounded by k circles denotes a multiple node consisting of the function value and all derivatives of order up to and including k. We also give plots of the cardinal function of the functional $A'_1 A''_1(f) = f(0, 0)$.

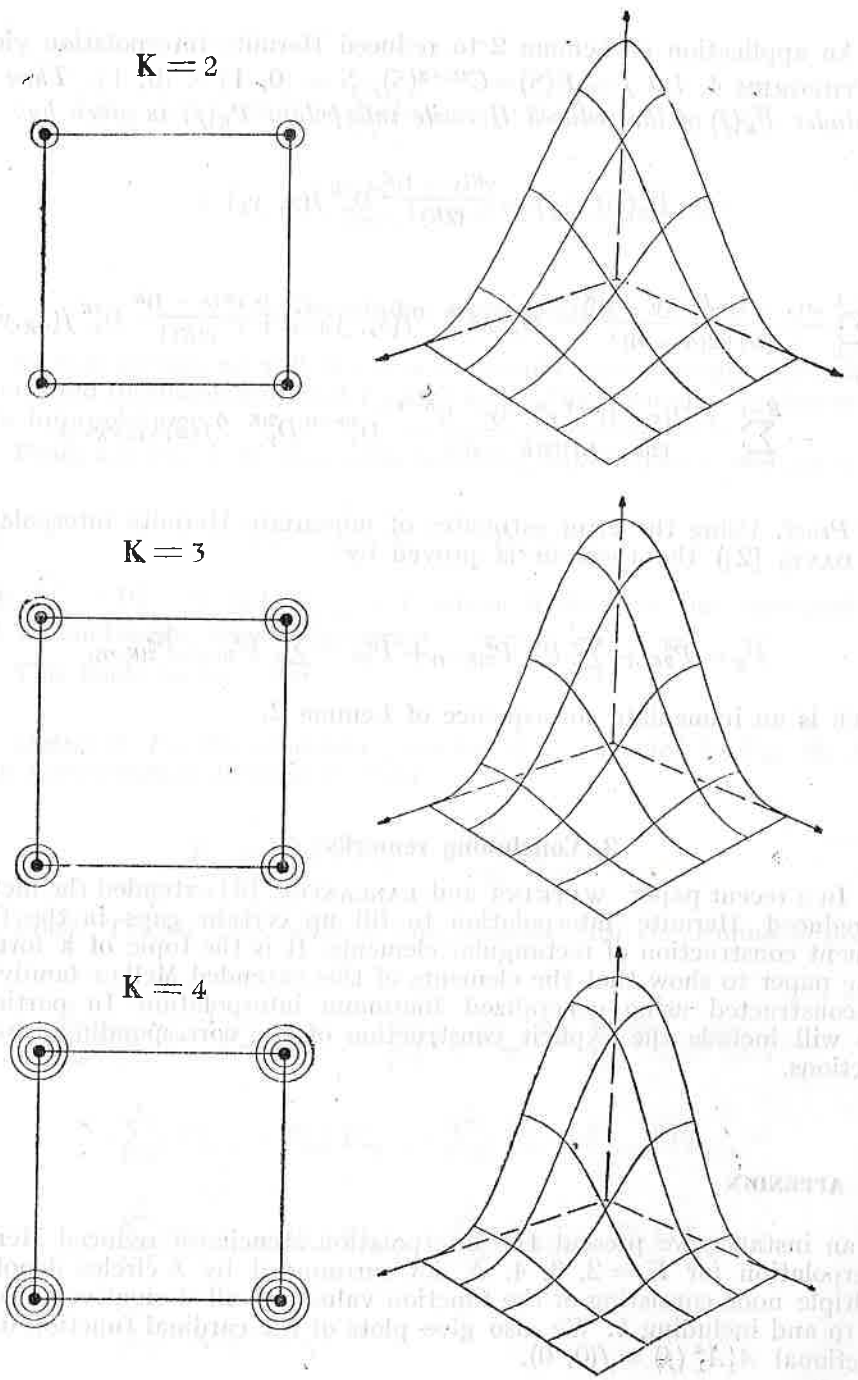

Fig. 1

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Dr. F. J. DELVOS
Lehrstuhl für Mathematik I
University of Siegen
Hölderlinstr. 3
D–5900 Siegen 21
West Germany

Dr. H. POSDORF
Rechenzentrum
University of Bochum
Universitätsstr. 150
D–4630 Bochum
West Germany