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CONES IN A CONVEXITY SPACE; ORDERED CONVEXITY SPACES

by

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1. In this paper we define the notion of cone in a convexity space (in the sense of V. W. BRYANT and R. J. WEBSTER [2], [3], [4], [5]) and we study the properties of some types of cones. Also, we define the concept of ordered convexity space and, using the cones theory in a convexity space, we give an example of ordered convexity space in the sense of the definition we'll give. This order relation is similar to the order relation induced by a convex cone in a Banach space.

2. Preliminaries

We denote, as in [2], by A, B, ... sets, by a, b, ... both the elements of a set and the singleton sets. Thus we use, as in [2], the sign $\subset$ instead of $\in$ except when a set is a member of a family of sets. The notation $A \approx B$ means $A \cap B \neq \emptyset$ and $(a, b, c, \ldots)$ stands for the set formed by the elements $a, b, c, \ldots$.

Let X be a nonempty set. We endow X with an operation $\cdot : X \times X \to 2^X$, called the product or join of a and b when $a, b \subset X$, and the inverse operation $/ : X \times X \to 2^X$, defined by $a/b = (x \subset X : a \subset bx)$, for all $a, b \subset X$. The couple $(X, .)$ is called a convexity space if it satisfies the following axioms:

(i) $ab \neq \emptyset,\ a/b \neq \emptyset$;
(ii) $a(bc) = (ab)c$;
(iii) $a/b \approx c/d \Rightarrow ad \approx bc$;
(iv) $aa = a = a/a$;
(v) $ab \approx ac \Rightarrow b = c$ or $b \approx ac$ or $c \approx ab$,

for any $a, b, c, d \subset X$. Here ab denotes $a \cdot b$.

For $a, b \subset X$, ab means the open line segment having its ends in a and in b, and a/b the half-line having the origin in a and not containing b.

The product and its inverse are extended to subsets of X by defining

$$AB = \bigcup_{\substack{a\subset A\\ b\subset B}} ab, \qquad A\,/\,B = \bigcup_{\substack{a\subset A\\ b\subset B}} a\,/\,b$$

A set $A \subset X$ is said to be convex if $AA \subset A$ and linear if $A\,/\,A \subset A$. If $A \subset X$ we denote by $[A]$ the convex hull of A, i.e. the intersection of all convex sets which contain A and by $\{A\}$ the linear hull of A, i.e. the intersection of all linear sets which contain A. For example, $[a, b] = ab \cup a \cup b$, $\{a, b\} = a\,/\,b \cup a \cup ab \cup b \cup b\,/\,a$. $[a, b]$ is the closed line segment joining a and b and $\{a, b\}$ is the line which contains a and b.

The concepts of independent set, basis, dimension for a convexity space are defined in the natural way, as in liniar spaces (see [4]).

A convexity space $(X, .)$ is said to be complete if whenever $A \subset X$ is convex, $a \subset A$ and $b \not\subset A$, there exists $c \subset [a, b]$ such that $ac \subset A$ and $bc \subset X \setminus A$. A set $H \subset X$, $H \neq \emptyset$, $H \neq X$, is a hyperplane in $(X, .)$ if H is linear and $\{H \cup x\} = X$ for any $x \subset X \setminus H$. The unordered pair (C, D), $C \subset X$, $D \subset X$, $C \neq \emptyset$, $D \neq \emptyset$ is said to be a convex pair in X if C, D are convex sets, $C \cap D = \emptyset$ and $C \cup D = X$. The reader may find in [3] many properties concerning the relation between hyperplanes and convex pairs.

A topological convexity space $(X, ., \tau)$ is a convexity space $(X, .)$ with a topology τ satisfying:

(it) $a \subset \overline{ab}$ for all $a, b \subset X$;

(iit) if $ab \approx U \in \tau$ then there exist $V, W \in \tau$ with $a \subset V$, $b \subset W$ and such that $a'b' \approx U$ whenever $a' \subset V$, $b' \subset W$;

(iiit) if $a/b \approx U \in \tau$ then there exist $V, W \in \tau$ with $a \subset V$, $b \subset W$ and such that $a'\,/\,b' \approx U$ whenever $a' \subset V$, $b' \subset W$.

3. Cones in a convexity space

Let $(X, .)$ be a convexity space.

D e f i n i t i o n 1. *Let θ be a point of X and $S \subset X$. We call cone generated by S and having the vertex in θ the set*

$$\mathcal{C}_0 S = (x \subset X : \exists y \subset S,\ x \subset y \cup \theta y \cup y\,/\,\theta \cup \theta)$$

In some places it is not necessary to put in evidence the set S which generates the cone. Thus we'll call cone a set K which was obtained as in definition 1, i.e. K is a cone if there exist a point $\theta \subset X$ and a set $S \subset X$ such that $K = \mathcal{C}_\theta S$. It is obvious that a cone K can be obtained using an infinity of sets S.

If $S \subset \mathbf{R}^n$, $\theta = 0$, $\mathcal{C}_\theta S$ is convex, closed and doesn't contain linear sets except the singleton sets then $\mathcal{C}_\theta S$ is a cone in sense of [9]. If $S \subset \mathbf{R}^n$, $\theta = 0$, and $\mathcal{C}_\theta S$ is convex then $\mathcal{C}_\theta S$ is a wedge in sense of [7].

D e f i n i t i o n 2. *A cone $K \subset X$ is said to be a generating cone if for every $x \subset X$ there exist $u, v \subset K$ such that $\{0, x\} \approx \{u, v\}$, where θ is the vertex of the cone K.*

In linear spaces this definition is equivalent to the definition of a generating cone given using the group structure of the space: if X is a Banach space then the convex cone $K \subset X$ (in sense of [9]) is a generating cone if for every $x \subset X$ there exist $u, v \subset K$ such that $x = u - v$. In order to proove this equivalence we suppose first that for every $x \subset X$ there exist $u, v \subset K$ such that $x = u - v$. Then, using the properties of vector operations in X we get that the line which contains the points u and v is parallel to the line which contains the origin of the space and the point x. Conversly, if $x \subset X$ and there exist $u, v \subset K$ such that $\{0, x\} \approx \{u, v\}$ than let $z \subset K$ such that $z/x \subset K$. Let $t \subset z/x$ and $y \subset K$ such that $\{t, y\} \approx \{0, x\}$ (this y exists because of the convexity of K). But on the line $\{t, y\}$ there exist a point s such that $\{0, s\} \approx \{x, z\}$. The points s and t satisfy the property that $x = t - s$.

D e f i n i t i o n 3. *The cone $K \subset X$ is a convex cone if $KK = K$.*

The theorems which follow are transpositions fo some classical results from the vectorial space case. Here is now a theorem of the Carathéodory type:

THEOREM 1. *If $(X, .)$ is a convexity space, dim $X = n$, $\theta \subset X$ and $S \subset X$ such that $\mathcal{C}_0 S$ is a convex cone and dim $\mathcal{C}_0 S = d$, then for every $x \subset \mathcal{C}_0 S$ there exist independent points $y_i \subset S$, $i = 1, 2, \ldots, p$, $p \leqq d$, such that $x \subset K$, where K is the intersection of all convex cones having the vertex in θ and including the points y_i, $i = 1, 2, \ldots, p$.*

Proof. $x \subset \mathcal{C}_0 S$ means that there exist $y \subset S$ such that $x \subset \theta y \cup y \cup y\,/\,\theta$. Applying for y the theorem 20 from [4] we get that there exist independent points, $y_1, y_2, \ldots, y_p \subset S$, $p \leqq d$, such that $y \subset [\theta, y_1, \ldots, y_p]$. If we put $K = \mathcal{C}_0\,[y_1, \ldots, y_p]$ then we get the cone required by theorem 1.

THEOREM 2. *If $(X, .)$ is a convexity space, τ a Hausdorff topology on X and $K \subset X$ a cone containing interior points (a full cone) then K is a generating cone.*

Proof. Let $x \subset X$ and $u \subset K$. This means that there exist a neighborhood V of the point u, $V \subset K$, $V \neq K$. Let $z \subset X$ such that $\{z, u\} \approx \{0, x\}$. Then there exist $v \subset \{z, u\} \cap V$. u and v are the points we looked for.

4. Ordered convexity spaces

D e f i n i t i o n 4. *An ordered convexity space $(X, ., \leqq)$ is a convexity space $(X, .)$ with an order relation, $\leqq$, satisfying:*

(io) *if $a \leqq b$ then for every $x \subset ab$, $a \leqq x \leqq b$;*

(iio) *if $a \leqq b$ then for every $x \subset a/b$ and $y \subset b/a$, $x \leqq a \leqq b \leqq y$;*

(iiio) *if a and b are not comparable then any $x, y \subset \{a, b\}$ are also not comparable.*

It's easy to verify that $(\mathbf{R}^n, \cdot, \leq)$, where $\cdot$ is defined by $a \cdot b = (x \subset \mathbf{R}^n : \exists\lambda,\ 0 < \lambda < 1,\ x = \lambda a + (1-\lambda)b)$ and $\leq$ is the order relation induced by the cone of the nonnegative elements from $\mathbf{R}^n$, is an ordered convexity space.

We remark that the axioms (io) and (iio) show that a linear set $\{a, b\}$, where $a \leq b$, in an ordered convexity space is isomorphic to a line containing two points which are comparable by the order relation induced by the cone of nonnegative elements from $\mathbf{R}^n$ or to a part of such a line.

THEOREM 3. *If $(X, .)$ is a convexity space and $K \subset X$ is a convex cone which doesn't contain linear sets (except the singleton sets) then the binary relation $\leq_K$ defined on X by:*

1° $x \leq_K x$ for any $x \subset X$;

2° $x, y \subset X$, $x \leq_K y$ if there exist $z \subset y/x$ such that $z/y \subset K$, is a partial order on X, and $(X, ., \leq_K)$ is an ordered convexity space.

Proof. 1° implies that $\leq_K$ is reflexive.

Antisymmetry. Let $x, y \subset K$ with $x \leq_K y$, $y \leq_K x$ and $x \neq y$. Than $y/x \subset K$ and $x/y \subset K$. From the convexity of K it follows that $xy \subset K$, than we have $\{x, y\} = x \cup y \cup xy \cup x/y \cup y/x \subset K$ which is a contradiction with the hypotesis that K doesn't contain lines.

If $x, y \subset X \setminus K$ with $x \leq_K y$, $y \leq_K x$ and $x \neq y$ than there exist $z_1 \subset y/x$ and $z_2 \subset x/y$ such that $z_1/y \subset K$ and $z_2/x \subset K$. But

$$z_1/y = z_1/x \subset y/x \subset \{x, y\}, \qquad z_2/x = z_2/y \subset x/y \subset \{x, y\}.$$

Let $u \subset z_1/y$ and $v \subset z_2/x$. Than $uv \subset K$ because of the convexity of K. Since $z_2 \subset vx$, $z_1 \subset uy$, $y \subset z_1x$ and $x \subset z_2y$ we have $y \subset uxy$ and $x \subset vxy$ then $xy \subset uvxy$. This can take place only if $xy \subset uv$. But $xy \subset X \setminus K$ and $uv \subset K$, which is a contradiction with the convexity of K. The case $x \subset X$, $y \subset K$ is similar.

Transitivity. Let $x, y, z \subset K$, $x \leq_K y$, $y \leq_K z$. Than we have $y/x \subset K$, $z/y \subset K$. Than $(y/x)(z/y) \subset K$. If $y \subset \{z, x\}$ than $x \leq_K z$. If $y \not\subset \{z, x\}$ let $u \subset z/y$ and $v \subset y/x$. Hence $z \subset uy$ and $y \subset vx$ and we have $zy \cup uvxy = uvx$. This means that $z \subset uvx$ and than there exist $w \subset z/x \cap uv$. But $uv \subset (y/x)(z/y) \subset K$, than $w \subset K$. Because u and v was arbitrarily choosed on z/y respectively on y/x, by a convenient choice of these points we find that every $w \subset z/x$, $w \subset K$, and than $z/x \subset K$, hence $x \leq_K z$. If $x, y, z \subset X \setminus K$, $x \leq_K y$, $y \leq_K z$ than there exist $a \subset y/x$ and $b \subset z/y$ such that $a/y \subset K$ and $b/z \subset K$. A similar argument for a and b proves that $z/x \subset (y/x)(z/y)$ and hence there exist $c \subset z/x$ such that $c/z \subset K$, than we have also $x \leq_K z$.

We use analogous arguments for the cases $x, y \subset X \setminus K$, $z \subset K$ and $x \subset X \setminus K$, $y, z \subset K$. The axioms (io), (iio), (iiio) are now obvious.

Remark. If X is a Banach space and K a convex cone in X the order relation defined in theorem 3 is equivalent to: if $x, y \subset K$ or $x \subset X \setminus K$, $y \subset K$, $x \leq_K y$ if $y - x \subset K$, and if $x, y \subset X \setminus K$, $x \leq_K y$ if $y - x \subset K^\circ$.

LEMMA 1. *If $(X, \cdot, \tau)$ is a complete topological convexity space locally convex and Hausdorff, $K \subset X$ a convex full cone which doesn't contain linear sets (except the singleton sets) and $x, y \subset X$ with $x \leq_K y$ than there exist two neighborhoods $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$ such that if $a \subset U$ and $b \subset V$ than either $a \leq_K b$ or a and b are not comparable by $\leq_K$.*

Proof. Let us suppose that $x, y \subset K$. Then let $K' \subset K$ a convex full cone such that $y \subset K'$, $x \not\subset \hat{K}'$ (see [2]) and $x \leq_{K'} y$. Since τ is Hausdorff and X is locally convex there exist a convex and open neighborhood $U \in \mathcal{V}(x)$ such that $U \cap K' = \emptyset$. Than, the theorem of Eidelheit type (see [1]) shows us that there exist a hyperplane $\mathbf{H}$ which separates the sets U and K'. Let $V \in \mathcal{V}(y) \cap \tau$, $a \subset U$ and $b \subset V$. Let $H = \hat{C} \cap \hat{D}$ where $\hat{C}$ and $\hat{D}$ are the closed half-spaces associated with H (see [3]) and let us suppose that $K' \subset \hat{C}$. Than $b/a \subset \hat{C}$ and we have either $b/a \subset K' \subset K$, $b/a \subset K$ or there exist a $z \subset b/a$ such that $z/a \subset \hat{C} \setminus K$. Hence we have either $a \leq_K b$ or a and b are not comparable by $\leq_K$.

If $x \subset X \setminus K$, $y \subset K$ we take $K' = K$ and we use the same reasoning.

If $x \subset X \setminus K$, $y \subset X \setminus K$ we take instead of K' the set $K'' = [y \cup K]$ and we use the same separating theorem and the same argument. The lemma is now completly prooved.

THEOREM 4. *If $(X, \cdot, \tau)$ is a complete topological convexity space, locally convex, Hausdorff, $K \subset X$ a convex cone which doesn't contain linear sets* (except the singleton sets), *$(x_n)_{n=1}^{\infty} \subset X$, $(y_n)_{n=1}^{\infty} \subset X$ with $x_n \leq_K y_n$, $n = 1, 2, \ldots$ and if $x_n \to x$ and $y_n \to y$ $(n \to \infty)$ than either $x \leq_K y$ or x and y are not comparable by $\leq_K$.*

Proof. Let us suppose that the assertion $x \leq_K y$ is false. Than either $y \leq_K x$ or x and y are not comparable by the order relation induced by K. Since the topology τ is Hausdorff and X locally convex than there exist $U \in \mathcal{V}(x)$, $V \in \mathcal{V}(y)$ convex and open such that $U \cap V = \emptyset$. Than, from [1] it follows that the sets U and V can be separated by a hyperplane H. Let $a \subset xy \cap H$. By the hypothesis it follows that there exist the convex and open neighborhoods $U' \in \mathcal{V}(x)$, $V' \in \mathcal{V}(y)$, $A \in \mathcal{V}(a)$ such that Lemma 1 and (iio) take place. This means that for any $u \subset U'$ and $v \subset V'$ we have either $v \leq_K u$ or v and u are not comparable by $\leq_K$. But since $x_n \to x$ and $y_n \to y$ when $n \to \infty$ it follows that there exist a $n_0 \subset N$ such that $x_n \subset U'$ and $y_n \subset V'$ for $n \geq n_0$. Hence for $n \geq n_0$ we have either $y_n \leq_K x_n$ or y_n and x_n are not comparable by $\leq_K$, which is a contradiction with the hypothesis $x_n \leq_K y$ for every $n \subset N$.

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NECESSARY OPTIMALITY CRITERIA IN NONLINEAR PROGRAMMING IN COMPLEX SPACE WITH DIFFERENTIABILITY

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In this paper we consider the problem

(P) *Minimize* $\operatorname{Re} f(\mathbf{z}, \bar{\mathbf{z}})$ *subject to* $\mathbf{z} \in X$, $\mathbf{g}(\mathbf{z}, \bar{\mathbf{z}}) \in S$,

where X *is a nonempty open set in* $\mathbf{C}^n$, S *is a polyhedral cone in* $\mathbf{C}^m$, $f: X \times \bar{X} \to C$ *and* $g: X \times \bar{X} \to \mathbf{C}^m$.

The paper is devided into four sections. In Section 1 notation is introduced and some preliminary results are given. In Section 2 we establish a necessary condition of the Fritz John type for Problem (P). In Section 3 seven kinds of complex constraint qualification (CCQ) are given and relations between them are established. In Section 4 we prove a Kuhn-Tucker type necessary condition for Problem (P).

1. Notation and Preliminary Results

Let $\mathbf{C}^n$ ($\mathbf{R}^n$) denote the n-dimensional complex (real) vector space with Hermitian (Euclidean) norm $||\cdot||$, $\mathbf{R}^n_+ = \{\mathbf{x}/\mathbf{x} = (x_j) \in \mathbf{R}^n,\ x_j \geq 0,\ j = 1, \ldots, n\}$ the non-negative orthant of $\mathbf{R}^n$, and $\mathbf{C}^{m\times n}$ the set of $m \times n$ complex matrices.

If $\mathbf{A}$ is a matrix or vector, then $\mathbf{A}^T$, $\bar{\mathbf{A}}$, $\mathbf{A}^H$ denote its transpose, complex conjugate and conjugate transpose respectively. For $\mathbf{z} = (z_j)$, $\mathbf{w} = (w_j) \in \mathbf{C}^n$; $\langle \mathbf{z}, \mathbf{w}\rangle = \mathbf{w}^H\mathbf{z}$ denotes the inner product of $\mathbf{z}$ and $\mathbf{w}$ and $\operatorname{Re} \mathbf{z} = (\operatorname{Re} z_j) \in \mathbf{R}^n$ denotes the real part of $\mathbf{z}$.