MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 9, N° 2, 1980, pp. 251–253

A CHARACTERISATION OF THE DIVIDED DIFFERENCES OF AN OPERATOR WHICH CAN BE REPRESENTED BY RIEMANN INTEGRALS

by

FLORIAN-ALEXANDRU POTRA

(București)

The notion of divided difference of an operator was introduced in 1956 by J. SCHRÖDER [8] and was used, five years later, by A. SERGEEV [10] to the extension of the secant method for the solution of nonlinear operator equations in Banach spaces.

DEFINITION 1. Let f be an operator having the domain of definition included into a Banach space $\mathcal{E}$ and taking values into a Banach space $\mathcal{F}$. Let x and y be two distinct points from the domain of definition of f. We shall call *a divided difference of f on the points x and y* a bounded linear operator $[x, y, f] : \mathcal{E} \to \mathcal{F}$ which satisfies the condition

$$[x, y\,; f](x - y) = f(x) - f(y). \tag{1}$$

In the above mentioned paper Sergeev supposed that the divided difference of f is symmetric (i.e. $[x, y\,; f] = [y, x\,; f]$) and that the divided difference of second order is bounded in norm.

In [11] and [12] U. ULM assumed that the divided difference of f satisfies only a Lipschitz condition of the form

$$\| [x, y\,; f] - [y, x\,; f] \| \leq H \| x - z \| \tag{2}$$

This assumption is weaker than the hypotheses of Sergeev but it implies however the symmetry of the divided difference (to see that take $z = x$ in (2)).

L. W. JOHNSON and D. R. SCHOLZ [3] considered that the divided difference of f satisfies the following Lipschitz condition

$$(3)\qquad \| [x, y ; f] - [u, v ; f] \| \leq H(\| x - u \| + \| y - v \|)$$

The above condition does not imply anymore the symmetry of the divided difference. In case the divided difference of f is symmetric then conditions (2) and (3) are equivalent. Condition (3) was also used in [4], [5] and [6]. A different Lipschitz condition was considered by J. W. SCHMIDT [7].

In what follows, when saying that the divided difference of f satisfies a Lipschitz condition we shall always mean condition (3). To be more precise let us state the following definition

DEFINITION 2. *Let $\mathcal{E}$ and $\mathcal{F}$ be two Banach spaces and let D be a convex subset of $\mathcal{E}$. We shall say that an operator $f: D \to \mathcal{F}$ has a Lipschitz divided difference on D if to each pair of distinct points $(x, y) \in D^2$ corresponds a bounded linear operator $[x, y; f] : \mathcal{E} \to \mathcal{F}$ which satisfes (1) and if inequality (3) holds for all $x, y, u, v \in D$, $x \neq y$, $u \neq v$.*

It is easy to see that if the operator f has a Lipschitz divided difference on D then f is Fréchet differentiable on D and the Fréchet derivative of f satisfies a Lipschitz condition of the form

$$(4)\qquad \| f'(x) - f'(y) \| \leq 2H \| x - y \|$$

Reversely if the operator f is Fréchet differentiable on D an if condition (4) is satisfied for all $x, y \in D$ then f has a Lipschitz divided difference on D. Indeed one can take

$$(5)\qquad [x, y ; f] = \int_0^1 f'(x + t(y - x))dt,$$

the above integral being in the sens of Riemann. The divided difference (5) was first considered by A. BELOSTOTZKII [9].

It is known that the divided difference of f is not unique (except when $\mathcal{E}$ is one-dimensional), so that it is important to characterize the divided differences of f of the form (5). In this respect we have obtained the following result:

THEOREM *Let $[.,. ; f]$ be a Lipschitz divided difference of f on D. The following two assertions are equivalent:*

(i) *equality (5) holds for every pair of distinct points $(x, y) \in D^2$;*

(ii) *for all $u, v \in D$ with $u \neq v$ and $2v - u \in D$ it is satisfied the relation*

$$(6)\qquad [u, v ; f] = 2[u, 2v - u ; f] - [v, 2v - u ; f]$$

Proof. By substituting $w = v - u$, relation (6) may be written under the form

$$(6')\qquad [u, u + w ; f] + [u + w, u + 2w ; f] = 2 [u, u + 2w ; f]$$

Then the implication (i) $\Rightarrow$ (ii) follows immediately observing that

$$\int_0^1 f'(u + tw)dt + \int_0^1 f'(u + w + tw)dt =$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{k=1}^{n} f'\left(u + \frac{k}{n} w\right) + \sum_{k=1}^{n} f'\left(u + w + \frac{k}{n} w\right)\right] =$$

$$= 2 \lim_{n\to\infty} \frac{1}{2n} \sum_{k=1}^{n} f'\left(u + \frac{k}{2n} 2w\right) = 2\int_0^1 f'(u + 2tw)\, dt$$

In order to prove the implication (ii) $\Rightarrow$ (i) let us first observe that from (6′) it follows that for every natural number n holds the relation

$$2^n [u, u + 2^n w ; f] = \sum_{k=1}^{2^n} [u + (k - 1)w, u + kw ; f]$$

Taking $u = x$ and $w = 2^{-n}(y - x)$ and using (3) we get the following inequality

$$\left\| [x, y ; f] - 2^{-n}\sum_{k=1}^{2^n} f'(x + k2^{-n}(y - x)) \right\| \leq 2^{-n+1} \| y - x \|$$

where from letting n to tend to infinity we infer the equality (5).

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Received 4. IX. 1980