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ON THE MAX-MIN NONLINEAR FRACTIONAL PROBLEM

by

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1. Introduction

In this note we consider the following max-min fractional problem:

PM1. *Find*

$$v = \max_{\mathbf{x}} \min_{\mathbf{y}} \frac{f(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})}$$

subject to :

$$h_i(\mathbf{x}, \mathbf{y}) \leqslant 0, \quad i = 1, 2, \ldots, m, \tag{1.1}$$

where $f: \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}$, $g: \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}$ *and* $h_i: \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}$ $(i = 1, 2, \ldots, m)$.

In the case when the functions f, g and h not depend effectivelly on $\mathbf{y}$, the problem PM1 is an usual fractional programming problem (see [1], [4], [5], [8]). If one suppose that the functions f, g and h are linear in respect to $\mathbf{x}$ and $\mathbf{y}$, then the problem PM1 is the linear fractional max-min problem considered in [2] and [9].

In the paper [4], B. MOND and B.D. CRAVEN give a method for solving the fractional programming problem using two nonfractional auxiliary problems. From this result it can be find similar results in the particular cases of the linear fractional programming [1], quadratic fractional programming [7] or polynomial fractional programming [6].

In this paper, we show that the result of MOND and CRAVEN [4] can be extended for the most general case of the max-min problem PM1.

2. The nonlinear fractional max–min problem

Next we follow the same way that in [4]. For this purpose, we consider the functions $E_i : \mathbf{R} \to \mathbf{R}$ $(i = 0, 1, \ldots, m)$ verifying the following two hypothesis:

i1) E_i $(i = 0, 1, \ldots, m)$ *are positive functions, that is:* $E_i(t) > 0, \ \forall t > 0$;

i2) E_0 *is a strictly increasing function.*

We denote:

$$F(\mathbf{u}, \mathbf{v}, t) = f\left(\frac{\mathbf{u}}{t}, \frac{\mathbf{v}}{t}\right) \cdot E_0(t),$$

(2.1) $$G(\mathbf{u}, \mathbf{v}, t) = g\left(\frac{\mathbf{u}}{t}, \frac{\mathbf{v}}{t}\right) \cdot E_0(t),$$

$$H_i(\mathbf{u}, \mathbf{v}, t) = h_i\left(\frac{\mathbf{u}}{t}, \frac{\mathbf{v}}{t}\right) \cdot E_0(t), \ i = 1, 2, \ldots, m.$$

We also suppose that there exist the limits:

$$\lim_{t\to 0} F(\mathbf{u}, \mathbf{v}, t) = F(\mathbf{u}, \mathbf{v}, 0),$$

$$\lim_{t\to 0} G(\mathbf{u}, \mathbf{v}, t) = G(\mathbf{u}, \mathbf{v}, 0),$$

$$\lim_{t\to 0} H_i(\mathbf{u}, \mathbf{v}, t) = H_i(\mathbf{u}, \mathbf{v}, t), \ i = 1, 2, \ldots, m.$$

Using the above notations, we associate to the problem PM1, the following non-fractional max–min problem:

PM2. *Find*

$$\max_{\mathbf{u}} \min_{(\mathbf{v}, t)} F(\mathbf{u}, \mathbf{v}, t)$$

subject to:

(2.2) $$G(\mathbf{u}, \mathbf{v}, t) = d,$$

(2.3) $$H_i(\mathbf{u}, \mathbf{v}, t) \leqslant 0, \ (i = 1, 2, \ldots, m), \ t \geqslant 0,$$

where $d \neq 0$ is a given real number.

We denote by:

(2.4) $$q(\mathbf{x}, \mathbf{y}) = \frac{f(\mathbf{x}, \mathbf{y})}{g(\mathbf{x}, \mathbf{y})},$$

the objective function of the max–min problem PM1.

Following the paper [2], the pair $(\mathbf{x}'', \mathbf{y}'') \in S = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^n \times \mathbf{R}^p : h_i(\mathbf{x}, \mathbf{y}) \leqslant 0, \ i = 1, 2, \ldots, m\}$, is called an *optimal solution* for the problem PM1 if the following two conditions are verified:

a1) $\min_{\mathbf{y}} \{q(\mathbf{x}'', \mathbf{y}) : \mathbf{y} \in S(\mathbf{x}'')\} = q(\mathbf{x}'', \mathbf{y}'')$;

a2) $q(\mathbf{x}'', \mathbf{y}'') \geqslant \min_{\mathbf{y}} \{q(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in S(\mathbf{x})\}, \ \forall \mathbf{x} \in P$,

where:

$$P = \{\mathbf{x} \in \mathbf{R}^n : \exists \mathbf{y} \in \mathbf{R}^p \text{ such that } h_i(\mathbf{x}, \mathbf{y}) \leqslant 0, \ i = 1, 2, \ldots, m\},$$

and

$$S(\mathbf{x}) = \{\mathbf{y} \in \mathbf{R}^p : h_i(x, y) \leqslant 0, \ i = 1, 2, \ldots, m\}.$$

Similarly are defined the optimal solution for the maximin problem PM2. The main result of the work is:

THEOREM 1. *If:*

(i) *the point* $(\mathbf{u}, \mathbf{v}, 0)$ *is not a feasible solution for the problem* PM2;

(ii) $0 < \operatorname{sign} d = \operatorname{sign} g(\mathbf{x}', \mathbf{y}')$ *for an optimal solution* $(\mathbf{x}', \mathbf{y}')$ *of the problem* PM1;

(iii) $(\mathbf{u}'', \mathbf{v}'', t'')$ *is an optimal solution of the problem* PM2;

then the pair $\left(\frac{\mathbf{u}''}{t''}, \frac{\mathbf{v}''}{t''}\right)$ *is an optimal solution of the problem* PM1.

Proof. From (i) it follows that $t'' > 0$, and then by i1), we obtain:

$$E_i(t'') > 0, \ i = 0, 1, \ldots, m.$$

But then, by (2.3) and (2.1) it results: $\left(\frac{\mathbf{u}''}{t''}, \frac{\mathbf{v}''}{t''}\right) \in S$. Let suppose that the pair $(\mathbf{u}''/t'', \mathbf{v}''/t'')$ is not an optimal solution for the problem PM1. Then, by (2.4), it follows:

(2.5) $$\min \{q(\mathbf{x}', \mathbf{y}) : \mathbf{y} \in S(\mathbf{x}')\} = q(\mathbf{x}', \mathbf{y}') > q(\mathbf{u}''/t'', \mathbf{v}''/t'') = = \min \{q(\mathbf{u}''/t'', \mathbf{y}) : \mathbf{y} \in S(\mathbf{u}''/t'')\}.$$

On the other side, from the condition (ii), there exists $\theta > 0$, such that:

$$g(\mathbf{x}', \mathbf{y}') = \theta d.$$

Taking:

$$t' = E_0^{-1}(1/\theta), \ \mathbf{u}' = t'\mathbf{x}', \ \mathbf{v}' = t'\mathbf{y}',$$

and using (2.1) — (2.3), it can be easily show that (u', v', t') is a feasible solution for the problem PM2. Also, by (2.1) and (2.2), we get:

$$\frac{f(x', y')}{g(x', y')} = \frac{F(u', v', t')}{G(u', v', t')} = \frac{F(u', v', t')}{d}, \tag{2.6}$$

$$\frac{f(u''/t'', v''/t'')}{g(u''/t'', v''/t'')} = \frac{F(u'', v'', t'')}{G(u'', v'', t'')} = \frac{F(u'', v'', t'')}{d}. \tag{2.7}$$

But, by (2.4)−(2.7), it follows the inequality:

$$F(u', v', t') > F(u'', v'', t''),$$

which is contrary to the assumption that (u'', v'', t'') is an optimal solution for the problem PM2. Hence $(u''/t'', v''/t'')$ is an optimal solution of the problem PM1, and the theorem is proven.

By Theorem 1, if the problem PM1 has an optimal solution, this solution can be obtained by solving the following two problems:

PM3. *Find*

$$v_1 = \max_{u} \min_{(v, t)} F(u, v, t)$$

subject to:

$$G(u, v, t) = 1,$$

$$H_i(u, v, t) \leqslant 0, \ (i = 1, 2, \ldots, m), \ t \geqslant 0.$$

PM4. *Find*

$$v_2 = \max_{x} \min_{(y, t)} (-F(u, v, t))$$

subject to:

$$-G(u, v, t) = 1,$$

$$H_i(u, v, t) \leqslant 0, \ (i = 1, 2, \ldots, m), \ t \geqslant 0.$$

From Theorem 1, it can be easily show that $v = \max(v_1, v_2)$.

3. The linear fractional case

Now we consider the linear fractional max−min problem:

PMF. *Find*

$$v = \max_{x} \min_{y} \frac{cx + dy + r}{fx + gy + s}$$

subject to:

$$Ax + By \leqslant b, \tag{3.1}$$

$$x \geqslant 0, \quad y \geqslant 0, \tag{3.2}$$

where $c \in R^n$, $f \in R^n$, $d \in R^p$, $g \in R^p$, $b \in R^m$, $r \in R$, $s \in R$ *and the matrix* A *and* B *with reals elements are given, and* $x \in R^n$, $y \in R^p$.

If we take:

$$E_i(t) = t \ (i = 0, 1, \ldots, m).,$$

then, to the problem PMF, we can associate the following linear max−min problem (see, the problem PM3):

PML. *Find*

$$v_1 = \max_{u} \min_{(v, t)} (cu + dv + rt)$$

subject to:

$$Au + Bv - bt \leqslant 0, \tag{3.3}$$

$$fu + gv + st = 1, \tag{3.4}$$

$$u \geqslant 0, \quad v \geqslant 0, \quad t \geqslant 0. \tag{3.5}$$

For the problem PMF we suppose:

H1) *the set* $S = \{(x, y) \in R^n \times R^p : Ax + By \leqslant b, \ x \geqslant 0, \ y \geqslant 0\}$, *is nonvoid and bounded;*

H2) $fx + gy + s > 0, \ \forall (x, y) \in S$.

From the theorem 1, one gets the following result:

THEOREM 2. *If the conditions* H1) *and* H2) *hold, and if* (u'', v'', t'') *is an optimal solution for the problem* PML, *then the pair* $\left(\frac{u''}{t''}, \frac{v''}{t''}\right)$ *is an optimal solution for the problem* PMF.

Proof. Because the conditions (ii) and (iii) of Theorem 1 are verified, it remains to show that $(u'', v'', 0)$ is not a feasible solution of the problem PML (the condition (i) of Theorem 1). That is, it must proved that every (u, v, t) verifying (3.3)−(3.5) has $t > 0$. In the contrar case, there exists a pair (u, v), such that:

$$Au + Bu \leqslant 0,$$

$$u \geqslant 0, \quad v \geqslant 0.$$

Then by (3.4), it follows that (u, v) is not the null vector. But if the pair (x, y) verifies (3.1) and (3.2), then the pair $(x + wu, y + wv)$ verifies also the inequalities (3.1), (3.2), for every $w \geqslant 0$, which is contrary to the boundness of the set S supposed by H1). This completly proofs the theorem.

In the end we make the remarque, that the linear max-min problem PML can be solved by the method given by J. E. FALK [3]. Also from Theorem 1, it can be obtained similar results with Theorems 2, for the max-min fractional problems with nonlinear fractional objective functions and linear constraints, such as homogenous or polinomial fractional objective functions.

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