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UPPER BOUNDS FOR THE LATENT ROOTS OF LAMBDA-MATRICES*

by

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We use matricial norms to determine upper bounds for the absolute values of the latent roots of lambda-matrix

$$\mathbf{A}(\lambda) = \mathbf{I}\lambda^n + \mathbf{A}_{n-1}\lambda^{n-1} + \ldots + \mathbf{A}_1\lambda + \mathbf{A}_0$$

whers $\mathbf{A}_l$ are s-quare complex matrices i.e. $\mathbf{A}_l \in \underset{s,s}{\mathbf{M}}(\mathbf{C})$, $(l = 0, 1, \ldots, n-1)$,

$\mathbf{I}$: unit matrix.

The latent roots λ of $\mathbf{A}(\lambda)$ are the zeros of det $\mathbf{A}(\lambda)$. It is known that if $\rho(\mathbf{A})$ is the spectral radius of a matrix $\mathbf{A}$, then [2, 3, 4] $\rho(\mathbf{A}) \leqslant \rho(\varphi(\mathbf{A}))$ and [1, 4] $\rho(\mathbf{A}) \leqslant [\rho(\varphi(\mathbf{A}^2))]^{1/2}$, where $\varphi(\mathbf{M})$ denotes a matricial norm of a matrix $\mathbf{M}$.

Denoting by $\mathbf{C}$ the block-companion matrix of $\mathbf{A}(\lambda)$; forming the matrix $\mathbf{C}^2$; and taking the (scalar) norm $\|\cdot\|_i$ [1] $(i = 1, \infty)$, of each block of

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1) For $\mathbf{B} = (\beta_{ij}) \in \underset{r,s}{\mathbf{M}}(\mathbf{C})$, we let

$$\|\mathbf{B}\|_1 := \underset{j=1,2,\ldots,s}{\mathrm{Max}} \left\{\sum_{i=1}^{r} |\beta_{ij}|\right\}, \quad \|\mathbf{B}\|_\infty := \underset{i=1,2,\ldots,r}{\mathrm{Max}} \left\{\sum_{j=1}^{s} |\beta_{ij}|\right\}$$

$C^2 \to$ so obtaining a matricial norm φ_i, $(i = 1, \infty)$ — we have the following relation between non-negative n-square matrices:

$$\varphi_i(\mathbf{C}^2) = \left[\begin{array}{c|cccc} \mathbf{0} & \|-\mathbf{A}_0\|_i & \|-\mathbf{A}_{-1} & + \ \mathbf{A}_0 & \mathbf{A}_{n-1}\|_i \\ & \|-\mathbf{A}_1\|_i & \|-\mathbf{A}_0 & + \ \mathbf{A}_1 & \mathbf{A}_{n-1}\|_i \\ \hline & \|-\mathbf{A}_2\|_i & \|-\mathbf{A}_1 & + \ \mathbf{A}_2 & \mathbf{A}_{n-1}\|_i \\ \|\mathbf{E}_{n-2}\|_i & \cdots & & \cdots & \\ & \|-\mathbf{A}_{n-1}\|_i & \|-\mathbf{A}_{n-2} & + \ \mathbf{A}_{n-1} & \mathbf{A}_{n-1}\|_i \end{array}\right] \leqslant$$

$$\leqslant \left[\begin{array}{c|c|ccc} \mathbf{0} & \mathbf{0} & \|\mathbf{A}_0\|_i & \|\mathbf{A}_{-1}\|_i + \|\mathbf{A}_0\|_i & \|\mathbf{A}_{n-1}\|_i \\ & & \|\mathbf{A}_1\|_i & \|\mathbf{A}_0\|_i + \|\mathbf{A}_1\|_i & \|\mathbf{A}_{n-1}\|_i \\ \hline \|\mathbf{E}_2\|_i & \mathbf{0} & \|\mathbf{A}_2\|_i & \|\mathbf{A}_1\|_i + \|\mathbf{A}_2\|_i & \|\mathbf{A}_{n-1}\|_i \\ & & \|\mathbf{A}_3\|_i & \|\mathbf{A}_2\|_i + \|\mathbf{A}_3\|_i & \|\mathbf{A}_{n-1}\|_i \\ \hline & & \|\mathbf{A}_4\|_i & \|\mathbf{A}_3\|_i + \|\mathbf{A}_4\|_i & \|\mathbf{A}_{n-1}\|_i \\ \mathbf{0} & \|\mathbf{E}_{n-4}\|_i & \cdots & \cdots & \\ & & \|\mathbf{A}_{n-1}\|_i & \|\mathbf{A}_{n-2}\|_i + \|\mathbf{A}_{n-1}\|_i & \|\mathbf{A}_{n-1}\|_i \end{array}\right] \in \underset{n,n}{M}(\mathbf{R}_+),\ i=1,\infty)$$

(with $\mathbf{E}_p := p$-bloc-unit matrix $:= \begin{pmatrix} \mathbf{I} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{I} \end{pmatrix} \in \underset{ps,ps}{\mathbf{M}}(\mathbf{C})$; $\mathbf{A}_{-1} := \mathbf{0}$).

Now we consider a partitioning in four blocks in such a way that both diagonal blocks are square and we take a (scalar) norm of each one of the four blocks — so obtaining again a matricial norm.

We get

(1) $$\rho(\varphi_i(\mathbf{C}^2)) \leqslant \rho \begin{pmatrix} \mathbf{0} & \alpha_i \\ \mathbf{1} & \beta_j \end{pmatrix}, \quad (i,\ j = 1,\ \infty)$$

where, for $n > 3$[2],

$$\alpha_j := \underset{(i,j=1,\infty)}{\left\| \begin{array}{c|ccc} \mathbf{0} & \|\mathbf{A}_0\|_i & \|\mathbf{A}_{-1}\|_i + \|\mathbf{A}_0\|_i & \|\mathbf{A}_{n-1}\|_j \\ & \|\mathbf{A}_1\|_i & \|\mathbf{A}_0\|_i + \|\mathbf{A}_1\|_i & \|\mathbf{A}_{n-1}\|_i \end{array} \right\|_j} \in \underset{2,n-2}{\mathbf{M}}(\mathbf{R}_+)$$

[2]) For $n = 2$ and $n = 3$, we see easily what the expressions of α_j e β_j are.

and

$$\beta_j := \underset{(i,j=1,\infty)}{\left\| \begin{array}{c|cccc} \mathbf{0} & \|\mathbf{A}_2\|_i & \|\mathbf{A}_1\|_i & + \ \|\mathbf{A}_2\|_i & \|\mathbf{A}_{n-1}\|_i \\ & \|\mathbf{A}_3\|_i & \|\mathbf{A}_2\|_i & + \ \|\mathbf{A}_3\|_i & \|\mathbf{A}_{n-1}\|_i \\ \hline & \|\mathbf{A}_4\|_i & \|\mathbf{A}_3\|_i & + \ \|\mathbf{A}_4\|_i & \|\mathbf{A}_{n-1}\|_i \\ \|\mathbf{E}_{n-4}\|_i & \cdots & & \cdots & \\ & \|\mathbf{A}_{n-1}\|_i & \|\mathbf{A}_{n-2}\|_i & + \ \|\mathbf{A}_{n-1}\|_i & \|\mathbf{A}_{n-1}\|_i \end{array} \right\|} \in \underset{n-2,n-2}{\mathbf{M}}(\mathbf{R}_+)$$

In particular, we have

$$\alpha_1 = \operatorname{Max}\ \{\|\mathbf{A}_0\|_i + \|\mathbf{A}_1\|_i,\ \|\mathbf{A}_0\|_i\ \|\mathbf{A}_{n-1}\| + \|\mathbf{A}_0\|_i + \|\mathbf{A}_1\|_i\ \|\mathbf{A}_{n-1}\|_i\}, \quad (i = 1,\ \infty)$$

and

$$\beta_1 = \operatorname{Max} \left\{1,\ \sum_{l=2}^{n-1} \|\mathbf{A}_l\|_i,\ \sum_{l=2}^{n-1} (\|\mathbf{A}_{l-1}\|_i + \|\mathbf{A}_l\|_i\ \|\mathbf{A}_{n-1}\|_i)\right\} \quad (i = 1,\ \infty)$$

which are not difficult to obtain.

As we can find exactly the eigenvalues of the non-negative 2-square matrix in relation (1), and remembering that the latent roots λ are the eigenvalues of $\mathbf{C}$, we obtain finally

(2) $$|\lambda| \leqslant \left(\frac{\beta_j + \sqrt{\beta_j^2 + 4\,\alpha_j}}{2}\right)^{1/2}, \quad j = 1,\ \infty),$$

an easily calculable bound.

Numerical Example. Let us take

$$\mathbf{A}(\lambda) = \mathbf{I}\lambda^6 + \mathbf{A}_5\lambda^5 + \mathbf{A}_4\lambda^4 + \mathbf{A}_3\lambda^3 + \mathbf{A}_2\lambda^2 + \mathbf{A}_1\lambda + \mathbf{A}_0,$$

with

$$\mathbf{A}_0 = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix},\ \mathbf{A}_1 = \begin{bmatrix} -1 & 7 \\ 2 & 3 \end{bmatrix},\ \mathbf{A}_2 = \begin{bmatrix} 9 & 0 \\ 3 & -4 \end{bmatrix},\ \mathbf{A}_3 = \begin{bmatrix} 11 & -31 \\ 0 & 2 \end{bmatrix},$$

$$\mathbf{A}_4 = \begin{bmatrix} 3 & -14 \\ 7 & 9 \end{bmatrix},\ \mathbf{A}_5 = \begin{bmatrix} 3 & -1 \\ 2 & -6 \end{bmatrix}.$$

Taking the (scalar) norm $\|.\|_1$, we have

$$\alpha_1 = 118 \quad \text{and} \quad \beta_1 = 603.$$

And we obtain $|\lambda| \leqslant 24{,}57$

Remark

The bound given by (2) can be better than other known ones. For example:

i) taking [5] the relation

$$|\lambda| \leqslant \frac{1}{2}\,\rho\begin{pmatrix}1+\|C\|_\infty & \gamma_\infty \\ 1 & \|A_{n-1}\|_\infty + \|C\|_\infty\end{pmatrix}, \text{ where } \gamma_\infty := \left\|\begin{matrix}A_0\\ \vdots \\ A_{n-2}\end{matrix}\right\|_\infty$$

we obtain $|\lambda| \leqslant 27{,}50$;

ii) taking [6] the relation

$$\|\lambda\| \leqslant \left(\frac{1+\sum_{l=0}^{n-1}\|A_l^*\|_1\,\|A_l\|_1 + \sqrt{\left(1+\sum_{l=0}^{n-1}\|A_l^*\|_1\,\|A_l\|_1\right)^2 - 4\,\|A_0^*\|_1\,\|A_0\|_1}}{2}\right)^{1/2}$$

we get $|\lambda| \leqslant 45{,}30$.

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