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ON APPROXIMATION OF CONTINOUS FUNCTIONS IN THE METRIC $\int_0^1 |x(t)|\,dt$

by

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1. Introduction. In 1924 JACKSON [3] published a paper on the linear approximation of continuous functions in the mean. One central result is that the best approximation from the space of ordinary polynomials is unique. A generalization to the approximation from a Haar space has been published in 1958 by PTAK [5]. In this paper we present an extension of these theorem for the case of nonlinear approximation with constraints.

Let T and E be normed linear spaces and P a subset of E, $F : E \to T$ mapping and let $F[P]$ be the set of approximating functions. If $w \in T$ then a best approximation to w is an element $v_0 \in F[P]$ with $\|w - v_0\| \leqslant \|w - v\|$ for every $v \in F[P]$. Here we shall study the question if such a best approximation is unique. Let P be given as a set described by inequality and equality constraints: I is a finite set, $(\Gamma_j)_{j\in I}$ is a set of compact topological spaces, P_0 is an open subset of E, Z is a Banach space and let mappings $g_{\tau,j} : E \to \mathbf{R}(\tau \in \Gamma_j,\ j \in I)$ and $p : E \to Z$ be given. Let

$$P = \bigcap_{j\in I} \bigcap_{\tau\in\Gamma_j} \{\alpha \in E \mid g_{\tau,j}(\alpha) \leqslant 0\} \cap \{\alpha \in E \mid p(\alpha) = \theta\} \cap P_0.$$

We assume that E is complete, F is Fréchet-differentiable at every $\alpha \in P$, the sets $(g_{\tau,j})_{\tau\in\Gamma_j}$ $(j \in I)$ have the property D1 at every $\alpha \in P$ (see WARTH [6]) and p has the property D2 at every $\alpha \in P$) (see WARTH [6])*.

* D_1 and D_2 are differentiability assumptions.

For $\alpha \in P$ let

$$I(\alpha) = \{j \in I \mid \max_{\tau \in \Gamma_j} g_{\tau,j}(\alpha) = 0\},\ \Gamma_j(\alpha) = \{\tau \in \Gamma_j \mid g_{\tau,j}(\alpha) = 0\}\ (j \in I)$$

$E(\alpha) = \{h \in E \mid p'(\alpha)h = 0,\ g'_{\tau,j}(\alpha)h \leqq 0$ for every $\tau \in \Gamma_j(\alpha)$ and $j \in I(\alpha)\}$ and $T(\alpha) = F'(\alpha)[E(\alpha)]$. Let $d(\alpha)$ denote the dimension of span $T(\alpha)$. $\alpha \in P$ is called regular, if $p'(\alpha)$ is surjective and there is a $h \in E$ with $p'(\alpha)h = 0$ and $g'_{\tau,j}(\alpha)h < 0$ for every $\tau \in \Gamma_j(\alpha)$ and $j \in I(\alpha)$. P is regular if every $\alpha \in P$ is regular.

In WARTH [6] it has been proved (local KOLMOGOROFF condition):

THEOREM 1. *If $v_0 = F(\alpha_0)$ is a best approximation to $w \in T$ and $\alpha_0 \in P$ is regular, then*

(1.1) *for every* $h \in E(\alpha_0)$ $\min \{l \circ F'(\alpha_0)h \mid l \in \sum_{w-v_0}\} \leqslant 0.$

$\sum_{w-v_0}$ *is the set of continuous, linear functionals l on T with* $l(w - v_0) = \|w - v_0\|$.

In this paper we suppose T to be the space $C[0, 1]$ of continuous, real-valued functions defined on the intervall $[0,1]$ with the integral norm. In this case (1.1) is equivalent to

(1.2) for every $q \in T(\alpha_0)$, $\displaystyle\int_{U^+_{w-v_0}} q(x)d\mu(x) - \int_{U^-_{w-v_0}} q(x)d\mu(x) \leqslant \int_{Z(w-v_0)} |q(x)|d\mu(x),$

with $U^+_{w-v_0} = \{x \in [0, 1] \mid w(x) - v_0(x) > 0\}$,

$$U^-_{w-v_0} = \{x \in [0, 1] \mid w(x) - v_0(x) < 0\}$$

and $Z(w - v_0) = \{x \in [0, 1] \mid w(x) - v_0(x) = 0\}$.

v_0 is a best approximation to w if

(1.3)

$$\text{for every } v \in F[P], \int_{U^+_{w-v_0}} (v(x) - v_0(x))d\mu(x) - \int_{U^-_{w-v_0}} (v(x) - v_0(x))d\mu(x) \leqslant$$

$$\leqslant \int_{Z(w-v_0)} |v(x) - v_0(x)|d\mu(x)$$

holds (KOLMOGOROFF criterion).

$F[P]$ is called an α-sun (see BROSOWSKI, WEGMANN [1]) if $w \in T$ and v_0 is a best approximation to w, then v_0 is a best approximation to $v_0 + \lambda(f - v_0)$ for every $\lambda > 0$.

If $F[P]$ is an α-sun, $w \in T$ and v_0 is a best approximation to w, then (1.3) holds.

2. The Main Theorem. A L_1-signature (U^+, U^-) is a pair of disjoint, open subsets of $[0,1]$ with $U^+ \cup U^- \neq \emptyset$. Let $U = (U^+, U^-)$ and

$$\varepsilon_U(x) = \begin{cases} 1 & x \in U^+ \\ 0 \quad \text{if} & x \in Z(U) \\ -1 & x \in U^- \end{cases}$$

with $Z(U) = [0, 1] \setminus (U^+ \cup U^-)$.

The L_1*-signature is called extremal for $v_0 \in F[P]$, if for every $v \in F[P]$,*

$$\int \varepsilon_U(x)(v(x) - v_0(x))\, d\mu(x) \leqq \int_{Z(U)} |(x) - v_0| d\mu(x).$$

If $w \in C(X)$, $w \neq 0$, then the L_1-signature $U_w = (U^+_w, U^-_w)$ is defined by $U^+_w = \{x \in [0, 1] \mid w(x) > 0\}$, $U^-_w = \{x \in [0, 1] \mid w(x) < 0\}$.

If U und U_1 are L_1-signatures then $U_1 \supset U$ if $U^+_1 \supset U^+$ and $U^-_1 \supset U^-$.

Lemma 2. *Let $\alpha_0 \in P$ be regular. If U is a* L_1*-signature which is extremal for $v_0 = F(\alpha_0)$, then for every $q \in T(\alpha_0)$,*

$$\int \varepsilon_U(x)\, q(x)\, d\mu(x) \leqq \int_{Z(U)} |q(x)|\, d\mu(x).$$

Proof. $N = [0, 1] \setminus (U^+ \cup U^-)$ is closed. There are functions ε_1, $\varepsilon_2 \in C[0, 1]$ with

$$\varepsilon_1(x) = \begin{cases} 0 & x \in N \cup U^+ \\ & \text{if} \\ <0 & x \in U^- \end{cases} \qquad \varepsilon_2(x) = \begin{cases} 0 & x \in N \cup U^- \\ & \text{if} \\ >0 & x \in U^+ \end{cases}$$

since $N \cup U^+$ and $N \cup U^-$ are closed. Let $w = v_0 + \varepsilon_1 + \varepsilon_2$. Then $U^+_{w-v_0} = U^+$, $U^-_{w-v_0} = U^-$ and since U is extremal for v_0, by the KOLMOGOROFF criterion v_0 is a best approximation to w. Then since α_0 is regular, by theorem 1 the results follows. Using an idea of BROSOWSKI, WEGMANN [1, „Durchschnittsatz"] we obtain

Lemma 3. *Let $F[P]$ be an α-sun. Let $w \in C[0, 1]$ and suppose v_0, $v_1 \in F[P]$ are best approximations to w. Then $U = (U^+, U^-)$, with*

$$U^+ = U^+_{w-v_0} \cup U^+_{w-v_1},\quad U^- = U^-_{w-v_0} \cup U^-_{w-v_1}$$

is a L_1*-signature which is extremal for v_0. Furthermore*

$$Z(U) \subset Z(v_1 - v_0) \text{ and } \int \varepsilon_U(x)(v_1(x) - v_0(x))\, d\mu(x) = 0.$$

Proof. Let $\lambda > 0$ and $w_\lambda = w + \lambda(w - v_1)$. Since $F[P]$ is an α-sun, v_1 is a best approximation to w_λ and $\|w_\lambda - v_1\| = (1 + \lambda)\|w - v_1\|$.

However $\| w_\lambda - v_0 \| \leqq \| w_\lambda - w \| + \| w - v_0 \| = \lambda \| w - v_1 \| + \| w - v_0 \| = (1 + \lambda) \| w - v_1 \|$ implies that v_0 is a best approximation to w_λ. Since $F[P]$ is an **α-sun** for every $v \in F[P]$

$$\int_{U^+_{w_\lambda - v_0}} (v(x) - v_0(x)\ d\mu(x) - \int_{U^-_{w_\lambda - v_0}} (v(x) - v_0(x))\, d\mu(x) \leqq \int_{Z(w_\lambda - v_0)} | v(x) - v_0(x) \,|\, d\mu(x).$$

i.e. $U_{w_\lambda} - v_0$ is extremal for v_0.

$$\| w_\lambda - v_0 \| = \int | (w - v_0) + \lambda(w - v_1) \,|\, d\mu,$$

$$(1 + \lambda) \| w - v_1 \| = \int | w - v_0 \,|\, d\mu + \lambda \int | w - v_1 \,|\, d\mu.$$

Then $\| w_\lambda - v_0 \| = \| w_\lambda - v_1 \| = (1 + \lambda) \| w - v_1 \|$ implies

$$\int | (w - v_0) + \lambda(w - v_1) \,|\, d\mu = \int | w - v_0 | d\mu + \lambda \int | w - v_1 | d\mu$$

and with $| (w(x) - v_0(x)) + \lambda\, (w(x) - v_1(x)) \,| \leqq | w(x) - v_0(x) | + \lambda \,|\, w(x) - v_1(x) \,|$ $\qquad (x \in [0,1])$

we obtain $(w(x) - v_0(x))\ (w(x) - v_1(x)) \geqq 0 \qquad (x \in [0,1])$.

Consequently $U = U_{w_\lambda - v_0}$. Since $Z(U) \cup Z(v_1 - v_0)$

$$\int_{U^+} (v_1 - v_0)\ d\mu - \int_{U^-} (v_1 - v_0)\ d\mu \leqq 0.$$

Substituting v_1 for v_0 we obtain that U is extremal for v_1, thus

$$\int_{U^+} (v_0 - v_1)\ d\mu - \int_{U^-} (v_0 - v_1)\ d\mu \leqq 0$$

and the lemma is proved.

Let $v_0 \in F[P]$ and U is a L_1-signature. (F, P) has the *property* L_1U at (v_0, U) if the following holds:

If $U_1 \supset U$ and if there is a $v \in F[P]$, $v \neq v_0$ with $Z(v - v_0) \supset Z(U_1)$ and $\int \varepsilon_{U_1} (v - v_0)\ d\mu = 0$, then there is a $\alpha \in P$, so that

i) $F(\alpha) \neq v_0$;

ii) $Z(F(\alpha) - v_0) \supset Z(U_1), \int \varepsilon_{U_1} (F(\alpha) - v_0)\ d\mu = 0$;

iii) *There is a $q \in T(\alpha)$ so that for almost every $x \in [0, 1]$,*

$$q(x) = \begin{cases} >0 & x \in U_1^+ \\ 0 \text{ if } & x \in Z(U_1) \\ <0 & x \in U_1^- \end{cases}$$

(F, P) has the property L_1U if it has the property at (v_0, U) for every $v_0 \in F[P]$ and every L_1-signature U.

THEOREM 4. *If P is regular, $F[P]$ is an α-sun, $w \in C[0, 1]$ and v_0 is a best approximation to w, then v_0 is the only best approximation if (F, P) has the property* L_1U at (v_0, U_{w-v_0}).

Proof. Suppose there is a $v \in F[P]$, $v \neq v_0$ which is a best approximation to w. By lemma 3 $U^+ = U^+_{w-v} \cup U^+_{w-v_0}$, $U^- = U^-_{w-v} \cup U^-_{w-v_0}$ defines a L_1-signature $U = (U^+, U^-)$ with $Z(U) \subset Z(v - v_0)$ and $\int \varepsilon_U (v - v_0)\ d\mu = 0$. Since $U \supset U_{w-v_0}$ and (F, P) has the property L_1U at (v_0, U_{w-v_0}), there is an $\alpha \in P$ with $F(\alpha) \neq v_0$, $Z(F(\alpha) - v_0) \supset Z(U)$, $\int \varepsilon_U\ (F(\alpha) - v_0)\ d\mu = 0$ and there is a $q_0 \in T(\alpha)$ with

$$q_0(x) = \begin{cases} >0 & x \in U^+ \\ 0 \text{ if } & x \in Z(U) \\ <0 & x \in U^- \end{cases}$$

for almost every $x \in [0, 1]$. But U is extremal for $F(\alpha)$. Together with lemma 2 we obtain a contradiction. As a corollary we obtain

THEOREM 5. *If P is regular, $F[P]$ is an α-sun and (F, P) has the property* L_1U, *then every $w \in C[0, 1]$ has at most one best approximation.*

3. Examples. Let H be a linear subspace of $C[0, 1]$ of finite dimension d and A a subset of $[0, 1]$. Then H is called a Haar space on A if every $v \in H$ has at most $d - 1$ zeros in A.

1) Suppose for every $v_0 \in F[P]$ there is an $\alpha_0 \in P$, with $F(\alpha_0) = v_0$, $T(\alpha_0)$ is a Haar space on $(0, 1)$ and every difference $v - v_0$ with $v \in F[P]$, $v \neq v_0$ has at most $d(\alpha_0) - 1$ zeros in $(0, 1)$. Then (F, P) has property L_1U.

Let $v_0 \in F[P]$ and U is a L_1-signature. Suppose there are $v \in F[P]$, $v \neq v_0$ and a L_1-signature $U_1 \supset U$ with $Z(v - v_0) \supset Z(U_1)$ and $\int \varepsilon_{U_1} (v -$

$- v_0)d\mu = 0$. Then $Z(U_1) \cap (0, 1)$ contains at most $d(\alpha) - 1$ zeros. By a theorem of KREIN (see KARLIN, STUDDEN [4]) there is a $q \in T(\alpha)$, which has the desired properties. So (F, P) has property L_1U at (v_0, U) (for every $v_0 \in F[P]$ and every L_1-signature U).

Together with theorem 5 we obtain the generalization of results of JACKSON [3] and PTAK [5].

2) Let $n \in \mathbf{N}$, $E = \mathbf{R}^{n+1}$, $u_0, \ldots, u_n \in C[0, 1]$ linear independent and let $F : E \to T$ be given by $(\alpha_0, \ldots, \alpha_n) \to \Sigma\alpha_i u_i$. Instead of $F(\alpha)$ let us write F_α.

Let $0 \leqq k_1 < \ldots < k_r \leqq n$ and $0 \leqq l_1 < \ldots < l_s \leqq n$ so that $\{k_1, \ldots, k_r\} \cap \{l_1 \ldots l_s\} = \emptyset$. Let $A, B \subset \{1, 2, \ldots, r\}$, $a_j \in \mathbf{R}$ $(j \in A)$ $b_j \in \mathbf{R}$ $(j \in B)$ and $c_1 \ldots c_s \in \mathbf{R}$. Let

$$P = \{(\alpha_0, \ldots \alpha_n) \in E \mid a_j \leqq \alpha_{k_j} \text{ for every } j \in A,\ b_j \geqq \alpha_{k_j} \text{ for every } j \in B,\ c_j = \alpha_{l_j}\ j = 1, 2, \ldots s\}.$$

Suppose $a_j < b_j$ if $j \in A \cap B$. If $\alpha \in P$ then

$$T(\alpha) = \{\Sigma\beta_i u_i \mid \beta_{k_j} \geqq 0 \text{ if } \alpha_{k_j} = a_j \text{ and } j \in A,\ \beta_{k_j} \leqq 0 \text{ if } \alpha_{k_j} = b_j \text{ and } j \in B,\ \beta_{l_j} = 0\ j = 1, 2, \ldots, s\}.$$

Then P is regular, $F[P]$ is convex, hence an α-sun.

Now suppose $u_i(x) = x^i$ for $x \in [0, 1]$ and $i = 0, 1, \ldots n$. Then

$$F[P] = \{P\alpha = \Sigma\alpha_i u_i \mid \bar{a}_j \leqq F_\alpha^{(k_j)}(0) \text{ for every } j \in A,\ \bar{b}_j \geqq F_\alpha^{(k_j)}(0) \text{ for every } j \in B,\ \bar{c}_j = F_\alpha^{(l_j)}(0)\ j = 1, 2, \ldots s\},$$

with $\bar{a}_j = k_j!\,a_j$ $(j \in A)$, $\bar{b}_j = k_j!\,b_j$ $(j \in B)$, $\bar{c}_j = k_j!\,c_j$ $(j = 1, 2, \ldots s)$ and for every $\alpha \in P$,

$$T(\alpha) = \{q = \Sigma\beta_i u_i \mid q^{(k_j)}(0) \geqq 0 \text{ if } F_\alpha^{(k_j)}(0) = \bar{a}_j \text{ and } j \in A,\ q^{(k_j)}(0) \leqq 0 \text{ if } F_\alpha^{(k_j)}(0) = \bar{b}_j \text{ and } j \in B,\ q^{(l_j)}(0) = 0\ j = 1, 2, \ldots s\}.$$

If $q^{(j)}(0) = 0$ for $q \in \text{span}\{u_0, \ldots, u_n\}$ and $j \in \{0, 1, \ldots, n\}$, then $q \in \text{span}(\{u_0, \ldots, u_n\} \setminus \{u_j\})$. Thus

$$\text{span}(\{u_0, \ldots, u_n\} \setminus \{u_{l_1}, \ldots, u_{l_s}, (u)_{k_j})_{j \in I_1(\alpha) \cup I_2(\alpha)}\}) \subset T(\alpha),$$

where $I_1(\alpha) = \{j \in A \mid F_\alpha^{(k_j)}(0) = \bar{a}_j\}$, $I_2(\alpha) = \{j \in B \mid F_\alpha^{(k_j)}(0) = \bar{b}\}$.

Let $r_1(\alpha)$ (resp. $r_2(\alpha)$) the number of elements of $I_1(\alpha)$ (resp. $I_2(\alpha)$). We assume $k_1 > 0$, $l_1 > 0$. Then $r_1(\alpha) + r_2(\alpha) + s < n + 1$ for every $\alpha \in P$. Let $m(\alpha) = (n + 1) - (r_1(\alpha) + r_2(\alpha) + s)$ $(\alpha \in P)$. Let $v_1, v_0 \in F[P]$, $v_1 \neq v_0$; $\hat{v} = F(\alpha) = \frac{1}{2}(v_1 + v_0)$. Then $\hat{v} - v_0$ has at most $m(\alpha) - 1$ zeros in $(0, 1)$ (see WARTH [7]). As above we can use KREIN'S theorem to show that (F, P) has the property L_1U. Applying theorem 5 we obtain that there is at most one best approximation to every element of $C[0, 1]$. In WARTH [7] this has been shown in the case of uniform approximation by a similar method.

3) We can apply theorem 4 to restricted range approximation problems as in DEVORE [2] and we obtain the same result concerning uniqueness. Regularity in this case has been proved in WARTH [6].

4. Concluding remarks

A compaison of this paper with WARTH [7] shows, that the L_1-approximation and the uniform approximation of continuous functions can be treated by similar theories. Still open is the problem if there are nonconvex α-suns in the case of L_1-approximation.

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