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NECESSARY OPTIMALITY CRITERIA IN NONLINEAR PROGRAMMING IN COMPLEX SPACE WITH DIFFERENTIABILITY

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In this paper we consider the problem

(P) *Minimize* $\operatorname{Re} f(\mathbf{z}, \overline{\mathbf{z}})$ *subject to* $\mathbf{z} \in X$, $\mathbf{g}(\mathbf{z}, \overline{\mathbf{z}}) \in S$,

where X *is a nonempty open set in* $\mathbf{C}^n$, S *is a polyhedral cone in* $\mathbf{C}^m$, $f: X \times \overline{X} \to C$ *and* $g: X \times \overline{X} \to \mathbf{C}^m$.

The paper is devided into four sections. In Section 1 notation is introduced and some preliminary results are given. In Section 2 we establish a necessary condition of the Fritz John type for Problem (P). In Section 3 seven kinds of complex constraint qualification (CCQ) are given and relations between them are established. In Section 4 we prove a Kuhn-Tucker type necessary condition for Problem (P).

1. Notation and Preliminary Results

Let $\mathbf{C}^n$ ($\mathbf{R}^n$) denote the n-dimensional complex (real) vector space with Hermitian (Euclidean) norm $||\cdot||$, $\mathbf{R}^n_+ = \{\mathbf{x}/\mathbf{x} = (x_j) \in \mathbf{R}^n,\ x_j \geqq 0,\ j = 1, \ldots, n\}$ the non-negative orthant of $\mathbf{R}^n$, and $\mathbf{C}^{m\times n}$ the set of $m \times n$ complex matrices.

If $\mathbf{A}$ is a matrix or vector, then $\mathbf{A}^T$, $\overline{\mathbf{A}}$, $\mathbf{A}^H$ denote its transpose, complex conjugate and conjugate transpose respectively. For $\mathbf{z} = (z_j)$, $\mathbf{w} = (w_j) \in \mathbf{C}^n$; $\langle \mathbf{z}, \mathbf{w}\rangle = \mathbf{w}^H\mathbf{z}$ denotes the inner product of $\mathbf{z}$ and $\mathbf{w}$ and $\operatorname{Re} \mathbf{z} = (\operatorname{Re} z_j) \in \mathbf{R}^n$ denotes the real part of $\mathbf{z}$.

If $\mathbf{x} = (x_j)$, $\mathbf{y} = (y_j) \in \mathbf{R}^n$, we consider

$$\mathbf{x} \leqq \mathbf{y} (\mathbf{x} < \mathbf{y}) \text{ iff } x_j \leqq y_j (x_j < y_j) \text{ for any } j \in \{1, \ldots, n\},$$

$$\mathbf{x} \leqslant \mathbf{y} \quad \text{iff } \mathbf{x} \leqq \mathbf{y} \text{ and } \mathbf{x} \neq \mathbf{y}.$$

If $X \subseteq \mathbf{C}^n$ then $\overline{X} = \{\mathbf{z} \in \mathbf{C}^n / \overline{\mathbf{z}} \in X\}$ and $-X = \{\mathbf{z} \in \mathbf{C}^n / -\mathbf{z} \in X\}$.

The nonempty set S in $\mathbf{C}^m$ is a polyhedral cone if it is an intersection of closed half-spaces in $\mathbf{C}^m$, each containing $\mathbf{0}$ in its boundary [3], i.e.

$$(1) \qquad S = \bigcap_{k=1}^{p} H_{\mathbf{u}_k}, \text{ where } H_{\mathbf{u}_k} = \{\mathbf{v} \in \mathbf{C}^m / \operatorname{Re} \langle \mathbf{v}, \mathbf{u}_k \rangle \geqq 0\}, \ k = \overline{1, p}.$$

The polar S^* of the nonempty set S in $\mathbf{C}^m$ is defined by

$$S^* = \{\mathbf{u} \in \mathbf{C}^n / \mathbf{v} \in S \Rightarrow \operatorname{Re} \langle \mathbf{u}, \mathbf{v} \rangle \geqq 0\}.$$

If $S = \bigcap_{k=1}^{p} H_{\mathbf{u}_k}$ is a polyhedral cone (1), then

$$\operatorname{int} S = \{\mathbf{v} \in \mathbf{C}^m / \operatorname{Re} \langle \mathbf{v}, \mathbf{u}_k \rangle > 0, \ k = 1, \ldots, p\}.$$

or equivalently,

$$\operatorname{int} S = \{\mathbf{v} \in \mathbf{C}^m / \mathbf{0} \neq \mathbf{u} \in S^* \Rightarrow \operatorname{Re} \langle \mathbf{v}, \mathbf{u} \rangle > 0\},$$

and $\operatorname{int} S = \emptyset$ iff $S^* \cap (-S^*) = \{0\}$, [6].

A nonempty set S in $\mathbf{C}^m$ is a closed convex cone iff $(S^*)^* = S$, [3].

Let X be an open set in $\mathbf{C}^n$ and let $\mathbf{z}^0 \in X$. A function $\mathbf{g} = (g_k(\mathbf{w}^1, \mathbf{w}^2)) : X \times \overline{X} \to \mathbf{C}^p$ is differentiable at $(\mathbf{z}^0, \overline{\mathbf{z}}^0) \in X \times \overline{X}$ if for all $\mathbf{z} \in X$:

$$(2) \qquad \mathbf{g}(\mathbf{z}, \overline{\mathbf{z}}) = \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) + [\nabla_{\mathbf{z}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T (\mathbf{z} - \mathbf{z}^0) + $$
$$+ [\nabla_{\overline{\mathbf{z}}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T (\overline{\mathbf{z}} - \overline{\mathbf{z}}^0) + |\mathbf{z} - \mathbf{z}^0| \, \mathbf{a}(\mathbf{z}, \mathbf{z}^0),$$

where

$$\nabla_{\mathbf{z}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) = \left(\frac{\partial g_k}{\partial w_j^1} (\mathbf{z}^0, \overline{\mathbf{z}}^0) \right)_{\substack{1 \leqq k \leqq p \\ 1 \leqq j \leqq n}} \in \mathbf{C}^{n \times p},$$

$$\nabla_{\overline{\mathbf{z}}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) = \left(\frac{\partial g_k}{\partial w_j^2} (\mathbf{z}^0, \overline{\mathbf{z}}^0) \right)_{\substack{1 \leqq k \leqq p \\ 1 \leqq j \leqq n}} \in \mathbf{C}^{n \times p},$$

and

$$(3) \qquad \lim_{\mathbf{z} \to \mathbf{z}^0} \mathbf{a}(\mathbf{z}, \mathbf{z}^0) = \mathbf{0}$$

DEFINITION 1. *Let X be a nonempty set in $\mathbf{C}^n$, let $\mathbf{z}^0 \in X$, and let S be a closed convex cone in $\mathbf{C}^m$. The function $\mathbf{g} : X \times \overline{X} \to \mathbf{C}^m$ is said to be*

a) *convex at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ with respect to S if*

$$(4) \qquad \left. \begin{array}{l} \mathbf{z} \in X \\ 0 \leqq \lambda \leqq 1 \\ (1 - \lambda)\mathbf{z}^0 + \lambda \mathbf{z} \in X \end{array} \right\} \Rightarrow \begin{array}{l} \lambda \mathbf{g}(\mathbf{z}, \overline{\mathbf{z}}) + (1 - \lambda) \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) - \mathbf{g}(\lambda \mathbf{z} + \\ + (1 - \lambda)\mathbf{z}^0, \lambda \overline{\mathbf{z}} + (1 - \lambda)\overline{\mathbf{z}}^0) \in S. \end{array}$$

If in addition X is open and $\mathbf{g}$ is differentiable at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$, then from (4) it follows that

$$\mathbf{g}(\mathbf{z}, \overline{\mathbf{z}}) - \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) - [\nabla_{\mathbf{z}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T (\mathbf{z} - \mathbf{z}^0) -$$
$$- [\nabla_{\overline{\mathbf{z}}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^m (\overline{\mathbf{z}} - \overline{\mathbf{z}}^0) \in S$$

for each $\mathbf{z} \in X$.

b) *pseudo-convex at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ with respect to S if X is open, $\mathbf{g}$ is differentiable at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ and for any $\mathbf{z} \in X$*

$$(5) \qquad [\nabla_{\mathbf{z}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T (\mathbf{z} - \mathbf{z}^0) + [\nabla_{\overline{\mathbf{z}}} \mathbf{g}(\mathbf{z}^0, \mathbf{z}^0)]^T (\overline{\mathbf{z}} - \overline{\mathbf{z}}^0) \in S \Rightarrow$$
$$\Rightarrow \mathbf{g}(\mathbf{z}, \overline{\mathbf{z}}) - \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) \in S.$$

c) *concave* (pseudo-concave) *at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ with respect to S if $\mathbf{g}$ is convex* (pseudo-convex) *at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ with respect to $-S$.*

d) *convex* (pseudo-convex, concave, pseudo-concave) *on $X \times \overline{X}$ with respect to S if X is convex and $\mathbf{g}$ is convex* (pseudo-convex concave, pseudo-concave) *at any $(\mathbf{z}, \overline{\mathbf{z}}) \in X \times \overline{X}$ with respect to S.*

e) *with convex* (pseudo-convex) *real part at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ with respect to a closed convex cone T in $\mathbf{R}^m$ if $\mathbf{g}$ is convex* (pseudo-convex), *at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ with respect to the closed convex cone $CT = \{\mathbf{v} \in \mathbf{C}^m / \operatorname{Re} \mathbf{v} \in T\} \subseteq C^m$.*

f) *with convex* (pseudo-convex) *real part on $X \times \overline{X}$ with respect to a closed convex cone T in $\mathbf{R}^m$ if X is convex and $\mathbf{g}$ is with convex* (pseudo-convex) *real part at any $(\mathbf{z}, \overline{\mathbf{z}}) \in X \times \overline{X}$ with respect to T.*

LEMMA 1. *Let $\mathbf{A} \in C^{p \times n}$, $\mathbf{B} \in \mathbf{C}^{q \times n}$ and $\mathbf{D} \in \mathbf{C}^{r \times n}$ be given matrices, with $\mathbf{A}$ being nonvacuous. Then exactly one of the following two systems has a solution :*

$$\operatorname{Re}(\mathbf{A}\mathbf{z}) > \mathbf{0}, \ \operatorname{Re}(\mathbf{B}\mathbf{z}) \geqq \mathbf{0}, \ \mathbf{D}\mathbf{z} = \mathbf{0}, \ \mathbf{z} \in \mathbf{C}^n,$$

$$\begin{cases} \mathbf{A}^H \mathbf{u} + \mathbf{B}^H \mathbf{v} + \mathbf{D}^H \mathbf{w} = \mathbf{0}, \ \mathbf{u} \in \mathbf{R}^p, \ \mathbf{v} \in \mathbf{R}^q, \ \mathbf{w} \in \mathbf{C}^r, \\ \mathbf{u} \geqslant \mathbf{0}, \ \mathbf{v} \geqq \mathbf{0}. \end{cases}$$

The *proof* is given in [7].

LEMMA 2. *Let X be a nonempty convex set in $\mathbf{C}^n$, let $\mathbf{f}_k : X \times \overline{X} \to \mathbf{C}^{m_k}$, $k = 1, 2, 3$ be vector functions having convex real part on $X \times \overline{X}$ with respect to $\mathbf{R}_+^{m_k}$ $k = 1, 2, 3$, and let $\mathbf{h} : \mathbf{C}^{2n} \to \mathbf{C}^p$ be a linear vector function.*

If the system :

$\operatorname{Re} \mathbf{f}_1(\mathbf{z}, \overline{\mathbf{z}}) < \mathbf{0}$, $\operatorname{Re} \mathbf{f}_2(\mathbf{z}, \overline{\mathbf{z}}) \leqslant \mathbf{0}$, $\operatorname{Re} \mathbf{f}_3(\mathbf{z}, \overline{\mathbf{z}}) \leqq \mathbf{0}$, $\mathbf{h}(\mathbf{z}, \overline{\mathbf{z}}) = \mathbf{0}$, *has no solution* $\mathbf{z} \in X$, *then there exist* $\lambda^k \in \mathbf{R}_+^{m_k}$, $k = 1, 2, 3$, *and* $\mu \in \mathbf{C}^p$ *such that*

$$(\lambda^1, \lambda^2, \lambda^3, \mu) \neq \mathbf{0}$$

and

$$\operatorname{Re}\left[\sum_{k=1}^{3} \langle \mathbf{f}_k(\mathbf{z}, \overline{\mathbf{z}}), \lambda^k \rangle + \langle \mathbf{h}(\mathbf{z}, \overline{\mathbf{z}}), \mu \rangle\right] \geqq 0,$$

for all $\mathbf{z} \in X$.

The *proof* is given in [7].

LEMMA 3. *Let $S = \bigcap_{k=1}^{p} H_{\mathbf{u}_k}$ be a polyhedral cone in $\mathbf{C}^m$, let $k \in \{1, \ldots, p\}$ be fixed, and let X be a nonempty convex set in $\mathbf{C}^n$. If $\mathbf{g} : X \times \overline{X} \to \mathbf{C}$ is concave on $X \times \overline{X}$ with respect to S, then the function $h_k : X \times \overline{X} \to \mathbf{C}$ defined by the formula*

$$h_k(\mathbf{z}, \mathbf{w}) = -\langle \mathbf{g}(\mathbf{z}, \mathbf{w}), \mathbf{u}_k \rangle \text{ for all } (\mathbf{z}, \mathbf{w}) \in X \times \overline{X},$$

has convex real part on $X \times \overline{X}$ with respect to $\mathbf{R}_+$.

The *proof* is given in [7].

2. A Fritz John Theorem in Complex Space

THEOREM 1. *Let X be a nonempty open set in $\mathbf{C}^n$ and let $\mathbf{z}^0 \in X$. Let $f : X \times \overline{X} \to \mathbf{C}$ and $\mathbf{g} : X \times \overline{X} \to \mathbf{C}^m$ be differentiable functions at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$, let $S = \bigcap_{k=1}^{p} H_{\mathbf{u}_k}$ be a polyhedral cone in $\mathbf{C}^m$ and let*

$$E = \{k \in \{1, \ldots, p\} \,/\, \operatorname{Re} \langle \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0), \mathbf{u}_k \rangle = 0\}.$$

(6) $I = \{k \in E \,/\, \mathbf{g}$ is pseudo-convex at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ with respect to $H_{\mathbf{u}_k}\}$

$$J = E \setminus I, \; L = \{1, \ldots, p\} \setminus E.$$

If $\mathbf{z}^0$ is a local minimum point of Problem (P), *then the system*

(7) $$\operatorname{Re}[\overline{\nabla_{\mathbf{z}} f(\mathbf{z}^0, \overline{\mathbf{z}}^0) + \nabla_{\overline{\mathbf{z}}} f(\mathbf{z}^0, \overline{\mathbf{z}}^0)}]^H \mathbf{z} < 0,$$

(8) $$\operatorname{Re}[\overline{\nabla_{\mathbf{z}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\mathbf{u}_k + \nabla_{\overline{\mathbf{z}}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\overline{\mathbf{u}}_k}]^H \mathbf{z} > 0, \quad k \in J,$$

(9) $$\operatorname{Re}[\overline{\nabla_{\mathbf{z}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\mathbf{u}_k + \nabla_{\overline{\mathbf{z}}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\overline{\mathbf{u}}_k}]^H \mathbf{z} \geqq \mathbf{0}, \quad k \in I,$$

has no solution $\mathbf{z} \in \mathbf{C}^n$.

Proof. Let $\mathbf{z}^0$ be a local minimum point of Problem (P), i.e. there exists $\tilde{\delta} > 0$ such that if

(10) $$B(\mathbf{z}^0; \tilde{\delta}) = \{\mathbf{z} \in \mathbf{C}^n \,/\, \|\mathbf{z} - \mathbf{z}^0\| < \tilde{\delta}\}$$

and

$$Y = \{\mathbf{z} \in X \,/\, \mathbf{g}(\mathbf{z}, \overline{\mathbf{z}}) \in S\},$$

then

(11) $$\operatorname{Re} f(\mathbf{z}, \overline{\mathbf{z}}) \geqq \operatorname{Re} f(\mathbf{z}^0, \overline{\mathbf{z}}^0) \text{ for all } \mathbf{z} \in B(\mathbf{z}^0; \tilde{\delta}) \cap Y.$$

We shal show that if $\mathbf{z}$ satisfies the system (7) — (9), then a contradiction arises. Let $\mathbf{z}$ be a solution of the system (7)—(9). Then, since X is open and $\mathbf{z}^0 \in X$, there exists a $\hat{\delta} > 0$ such that

(12) $$\mathbf{z}^0 + \delta\mathbf{z} \in X \text{ for all } 0 \leqq \delta < \hat{\delta}.$$

Since f and $\mathbf{g}$ are differentiable at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ from (2) we have that for all $\delta \in\,]0, \hat{\delta}[$

(13) $$f(\mathbf{z}^0 + \delta\mathbf{z}, \overline{\mathbf{z}}^0 + \delta\overline{\mathbf{z}}) = f(\mathbf{z}^0, \overline{\mathbf{z}}^0) + \delta\{[\nabla_{\mathbf{z}} f(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T \mathbf{z} + [\nabla_{\overline{\mathbf{z}}} f(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T \overline{\mathbf{z}}\} + \delta\,|\,|\mathbf{z}|\,|\, a_0(\mathbf{z}^0 + \delta\mathbf{z}, \mathbf{z}^0)$$

and

(14) $$\mathbf{g}(\mathbf{z}^0 + \delta\mathbf{z}, \overline{\mathbf{z}}^0 + \delta\overline{\mathbf{z}}) = \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) + \delta\{[\nabla_{\mathbf{z}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T \mathbf{z} + \nabla_{\overline{\mathbf{z}}} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T \overline{\mathbf{z}}\} + \delta\,\|\mathbf{z}\|\,\mathbf{a}(\mathbf{z}^0 + \delta\mathbf{z}, \mathbf{z}^0),$$

where

(15) $$\lim_{\delta \to 0} a_0(\mathbf{z}^0 + \delta\mathbf{z}, \mathbf{z}^0) = 0 \quad \text{and} \quad \lim_{\delta \to 0} \mathbf{a}(\mathbf{z}^0 + \delta\mathbf{z}, \mathbf{z}^0) = \mathbf{0}.$$

From (14) and (15) it follows that

(16) $$\mathrm{Re}\langle g(z_0 + \delta z, \overline{z}^0 + \delta\overline{z}), u_k\rangle = \mathrm{Re}\langle g(z^0, \overline{z}^0), u_k\rangle +$$
$$+ \mathrm{Re}\,\langle [\nabla_z g(z^0, \overline{z}^0)]^T z + [\overline{\nabla_{\overline{z}} g(z^0, \overline{z}^0)^T}]\overline{z}, u_k\rangle +$$
$$+ \delta\,\|z\|\,\mathrm{Re}\,\langle a(z^0 + \delta z, z^0), u_k\rangle$$

for all $k \in \{1, \ldots, p\}$ and $\delta \in]0, \hat{\delta}[$,

and

(17) $$\lim_{\delta\to 0} \langle \mathrm{Re}\,\langle a(z^0 + \delta z, z^0), u_k\rangle = 0 \text{ for all } k \in \{1, \ldots, p\}.$$

(i) Let $k \in J$. Then from (16), (17) and (8) we deduce that there exists $\delta_k \in]0, \hat{\delta}[$ such that

$$\mathrm{Re}\,\langle g(z^0 + \delta z, \overline{z}^0 + \delta\overline{z}), u_k\rangle \geqslant \mathrm{Re}\,\langle g(z^0, \overline{z}^0), u_k\rangle \text{ for all } \delta \in]0, \delta_k[.$$

Since $\mathrm{Re}\,\langle g(z^0, \overline{z}^0), u_k\rangle = 0$, $k \in J \subseteq E$, it follows that

(18) $$g(z^0 + \delta z, \overline{z}^0 + \delta\overline{z}) \in H_{u_k} \text{ for all } \delta \in]0, \delta_k[,\ k \in J.$$

(ii) Let $k \in I$. Since g is pseudo-convex at $(z^0, \overline{z}^0)$ with respect to H_{u_k}, $k \in I$, from (5) and (9) we infer that

$$\mathrm{Re}\,\langle g(z^0 + \delta z, \overline{z}^0 + \delta\overline{z}) - g(z^0, \overline{z}^0), u_k\rangle \geqq 0, \text{ for all } \delta \in]0, \hat{\delta}[$$

and hence

(19) $$g(z^0 + \delta z, \overline{z}^0 + \delta\overline{z}) \in H_{u_k}, \text{ for all } \delta \in]0, \hat{\delta}[\text{ and } k \in I.$$

(iii) Let $k \in L$. Then, from (16) and (17), it follows that there exists $\delta_k \in]0, \hat{\delta}[$ such that $\mathrm{Re}\,\langle g(z^0 + \delta z, \overline{z}^0 + \delta\overline{z}), u_k\rangle \geqq 0$ for $\delta \in]0, \delta_k[$, and hence

(20) $$g(z^0 + \delta z, \overline{z}^0 + \delta\overline{z}) \in H_{u_k} \text{ for } \delta \in]0, \delta_k[,\ k \in L.$$

(iv) From (13), (15) and (7) we deduce that there exists $\delta_0 \in]0, \hat{\delta}[$ such that

(21) $$\mathrm{Re}\, f(z^0 + \delta z, \overline{z}^0 + \delta\overline{z}) < \mathrm{Re}\, f(z^0, \overline{z}^0), \text{ for all } \delta \in]0, \delta_0[.$$

Let r be the minimum of all positive numbers $\tilde{\delta}$, $\hat{\delta}$, δ_0, $\delta_k (k \in I \cup L)$. Then, from (10), (12), (18)–(21) we have $z^0 + \delta z \in B(z^0; \tilde{\delta}) \cap Y$ and

$\mathrm{Re}\, f(z^0 + \delta z, \overline{z}^0 + \delta\overline{z}) < \mathrm{Re}\, f(z^0, \overline{z}^0)$ for all $\delta \in]0, r[$, which contradicts (11). Hence the system (7)–(9) has no solution z in $\mathbf{C}^n$.

THEOREM 2. *Let X be a nonempty open set in $\mathbf{C}^n$, let $z^0 \in X$, let $S = \bigcap_{k=1}^{p} H_{u_k}$ be a polyhedral cone in $\mathbf{C}^m$ with nonempty interior, let $f : X \times \overline{X} \to \mathbf{C}$ and $g : X \times \overline{X} \to \mathbf{C}^m$ be differentiable functions at $(z^0, \overline{z}^0)$ and let E, I and J be defined by* (6).

If z^0 is a local minimum point of Problem (P), *then there exist $\tau \in \mathbf{R}_+$ and $u \in \left(\bigcap_{k\in E} H_{u_k}\right)^* \subseteq S^*$* such that:

(I) *there exists $\lambda = (\lambda_k) \in \mathbf{R}^m_+$ with :*

a) $u = \sum_{k=1}^{p} \lambda_k u_k$,

b) $\lambda_k = 0$ for any $k \in \{1, \ldots, p\} \setminus E$.

c) if $u_J = \sum_{k\in J} \lambda_k u$, then $(\tau, u_J) \neq 0$,

(II) $\tau\nabla_z f(z^0, \overline{z}^0) + \tau\overline{\nabla_{\overline{z}} f(z^0, \overline{z}^0)} - \nabla_z g(z^0, \overline{z}^0)u - \overline{\nabla_{\overline{z}} g(z^0, \overline{z}^0)u} = 0$,

(III) $\mathrm{Re}\,\langle g(z^0, \overline{z}^0), u\rangle = 0$.

Proof. In view of Theorem 1, System (7)–(9) has no solution $z \in \mathbf{C}^n$. Then, by Lemma 1 the system

(22) $$\tau[\nabla_z f(z^0, \overline{z}^0) + \overline{\nabla_{\overline{z}} f(z^0, \overline{z}^0)}] -$$
$$- \sum_{k\in E} \mu_k [\nabla_z g(z^0, \overline{z}^0)u_k + \overline{\nabla_{\overline{z}} g(z^0, \overline{z}^0)u_k}] = 0$$

has a solution $(\tau, \mu_E) \geqslant 0$ with

(23) $$(\tau, \mu_J) \geqslant 0 \quad \text{and} \quad \mu_I \geqq 0,$$

where $\mu_E = (\mu_k)_{k\in E}$, $\mu_J = (\mu_k)_{k\in J}$ and $\mu_I = (\mu_k)_{k\in I}$.

Define

(24) $$u_I = \sum_{k\in I} \lambda_k u_k,\quad u_J = \sum_{k\in J} \lambda_k u_k \text{ and } u = \sum_{k=1}^{p} \lambda_k u_k,$$

where

$$\lambda_k = \begin{cases} \mu_k, & \text{if } k \in E \\ 0, & \text{if } k \in \{1, \ldots, p\} \setminus E. \end{cases}$$

We have that $\mathfrak{u} \in (\bigcap_{k\in E} H_{\mathfrak{u}_k})^* \subseteq S^*$, and from (22) and (24) follows (Ia), (Ib), (II) and (III). It remained to show that $(\tau, \mathfrak{u}_J) \neq 0$.

(i) If $\tau \neq \mathbf{0}$, we have that $(\tau, \mathfrak{u}_J) \neq \mathbf{0}$.

(ii) If $\tau = 0$, from (23) it follows that $\mu_J \geqslant \mathbf{0}$. If $\mathfrak{u}_J = \mathbf{0}$, the system $\mathfrak{u}_J = \sum_{k\in J} \mu_k \mathfrak{u}_k = \mathbf{0}$, $\mu_k \geqslant 0$ has a solution. By Lemma 1, the system $\operatorname{Re}\langle \mathbf{w}, \mathfrak{u}_k\rangle > 0$, $k \in J$ has no solution $\mathbf{w} \in \mathbf{C}^m$, which contradicts $\operatorname{int} S \neq \emptyset$. Consequently $\mathfrak{u} \neq \mathbf{0}$, hence $(\tau, \mathfrak{u}_J) \neq \mathbf{0}$.

COROLLARY 1. *If the conditions of Theorem* 2 *are fulfilled and* $J = \emptyset$, *then* $\tau \neq 0$.

3. Seven Kinds of Complex Constraint Qualification

DEFINITION 2. *Let* X *be a nonempty set in* $\mathbf{C}^n$ *and let* S *be a closed convex cone in* $\mathbf{C}^m$.

The function $\mathfrak{g} : X\times\overline{X} \to \mathbf{C}^m$ *which defines the set* $Y = \{\mathbf{z} \in X / \mathfrak{g}(\mathbf{z}, \overline{\mathbf{z}}) \in S\}$, *is said to satisfy :*

1°. *Slater's complex constraint qualification* (CCQ) *with respect to* Y [6] *if* $\operatorname{int} S \neq \emptyset$ *and there exists* $\mathbf{z}^1 \in X$ *such that* $\mathfrak{g}(\mathbf{z}^1, \overline{\mathbf{z}}^1) \in \operatorname{int} S$.

2°. *the strict* CCQ *with respect to* Y [6] *if* $\operatorname{int} S \neq \emptyset$ *and there exist two points* $\mathbf{z}^0, \mathbf{z}^1 \in Y$, $\mathbf{z}^0 \neq \mathbf{z}^1$ *and* $\lambda \in]0, 1[$ *such that* $\mathbf{z}(\lambda) = (1-\lambda)\mathbf{z}^0 + \lambda\mathbf{z}^1 \in X$ *and*

$$\mathfrak{g}[\mathbf{z}(\lambda), \overline{\mathbf{z}(\lambda)}] - (1-\lambda)\mathfrak{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) - \lambda\mathfrak{g}(\mathbf{z}^1, \overline{\mathbf{z}}^1) \in \operatorname{int} S.$$

3°. *Karlin's* CCQ *with respect to* Y *if there exists no* $\mathbf{v} \in S^*$, $\mathbf{v} \neq \mathbf{0}$ *such that*

$$\operatorname{Re}\langle \mathfrak{g}(\mathbf{z}, \overline{\mathbf{z}}), \mathbf{v}\rangle \leqq 0 \textit{ for all } \mathbf{z} \in X.$$

LEMMA 4. *Let* X *be a nonempty set in* $\mathbf{C}^n$, *let* $S = \bigcap_{k=1}^{p} H_{\mathfrak{u}_k}$ *be a polyhedral cone in* $\mathbf{C}^m$, *let* $\mathfrak{g} : X\times\overline{X} \to \mathbf{C}^m$, *and let* $Y = \{\mathbf{z} \in X / \mathfrak{g}(\mathbf{z}, \overline{\mathbf{z}}) \in S\}$.

(i) *If* $\mathfrak{g}$ *satisfies Slater's* CCQ *with respect to* Y, *then* $\mathfrak{g}$ *satisfies Karlin's* CCQ *with respect to* Y.

(ii) *If* $\mathfrak{g}$ *satisfies the strict* CCQ *with respect to* Y, *then* $\mathfrak{g}$ *satisfies Slater's* CCQ *with respect to* Y.

(iii) *If in addition* X *is convex,* $\operatorname{int} S \neq \emptyset$ *and* $\mathfrak{g}$ *is concave on* $X\times\overline{X}$ *with respect to* S, *then Karlin's* CCQ *and Slater's* CCQ *are equivalent.*

Proof. (i) Let $\mathfrak{g}$ satisfy Slater's CCQ. Then $\operatorname{int} S \neq \emptyset$ and there exists $\mathbf{z}^1 \in X$ such that $\mathfrak{g}(\mathbf{z}^1, \overline{\mathbf{z}}^1) \in \operatorname{int} S$. Now, if $\mathbf{v} \in S^*$ and $\mathbf{v} \neq \mathbf{0}$, it follows

that $\operatorname{Re}\langle \mathfrak{g}(\mathbf{z}^1, \overline{\mathbf{z}}^1), \mathbf{v}\rangle > 0$, and hence there exists no $\mathbf{v} \in S^*$, $\mathbf{v} \neq \mathbf{0}$ such that $\operatorname{Re}\langle \mathfrak{g}(\mathbf{z}, \overline{\mathbf{z}}), \mathbf{v}\rangle \leqq 0$ for all $\mathbf{z} \in X$.

(ii) If the strict CCQ with respect to Y is satisfied, then $\operatorname{int} S \neq \emptyset$ and there exist two points $\mathbf{z}^0, \mathbf{z}^1 \in Y$, $\mathbf{z}^0 \neq \mathbf{z}^1$ and $\lambda \in]0, 1[$ such that $\mathbf{z}(\lambda) = (1-\lambda)\mathbf{z}^0 + \lambda\mathbf{z}^1 \in X$ and $\operatorname{Re}\langle \mathfrak{g}[\mathbf{z}(\lambda), \overline{\mathbf{z}(\lambda)}] - (1-\lambda)\,\mathfrak{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) - \lambda\mathfrak{g}(\mathbf{z}^1, \overline{\mathbf{z}}^1), \mathbf{v}\rangle > 0$ for all $\mathbf{v} \in S^*$, $\mathbf{v} \neq \mathbf{0}$. Since $\mathbf{z}^0, \mathbf{z}^1 \in Y$, we have $\operatorname{Re}\langle \mathfrak{g}[\mathbf{z}(\lambda), \overline{\mathbf{z}(\lambda)}], \mathbf{v}\rangle > 0$ for all $\mathbf{v} \in S^*$, $\mathbf{v} \neq \mathbf{0}$, hence the point $\mathbf{z} = \mathbf{z}(\lambda) \in X$ has the property that $\mathfrak{g}(\mathbf{z}, \overline{\mathbf{z}}) \in \operatorname{int} S$.

(iii) Let $\mathfrak{g}$ satisfy Karlin's CCQ with respect to Y. If $\mathfrak{g}$ does not satisfy Slater's CCQ, then the system

$$-\operatorname{Re}\langle \mathfrak{g}(\mathbf{z}, \overline{\mathbf{z}}), \mathfrak{u}_k\rangle < 0,\ k \in \{1, \ldots, p\},$$

has no solution $\mathbf{z} \in X$. Then, by Lemmas 3 and 2, there exists $\lambda = (\lambda_k) \in \mathbf{R}^p$, $\lambda \geqslant 0$ such that $-\operatorname{Re}\langle \mathfrak{g}(\mathbf{z}, \overline{\mathbf{z}}), \mathbf{v}\rangle \geqq 0$ for all $\mathbf{z} \in X$, where $\mathbf{v} = \sum_{k=1}^{p} \lambda_k \mathfrak{u}_k$. Obviously, $\mathbf{v} \in S^*$. If $\mathbf{v} = \mathbf{0}$, the system $\sum_{k=1}^{p} \lambda_k \mathfrak{u}_k = \mathbf{0}$, $\lambda = (\lambda_k) \geqslant \mathbf{0}$ has a solution, hence by Lemma 1, the system $\operatorname{Re}\langle \mathbf{w}, \mathfrak{u}_k\rangle > 0$, $k = 1, \ldots, p$ has no solution $\mathbf{w} \in \mathbf{C}^n$, which contradicts $\operatorname{int} S \neq \emptyset$. Consequently $\mathbf{v} \neq \mathbf{0}$. Hence, there exists $\mathbf{v} \in S^*$, $\mathbf{v} \neq \mathbf{0}$ such that $\operatorname{Re}\langle \mathfrak{g}(\mathbf{z}, \overline{\mathbf{z}}), \mathbf{v}\rangle \leqq 0$ for all $\mathbf{z} \in X$, but this contradicts the fact that $\mathfrak{g}$ satisfies Karlin's CCQ. From this and (i) it follows that Slater's CCQ and Karlin's CQQ are equivalent.

DEFINITION 3. *Let* X *be an open set in* $\mathbf{C}^n$, *let* $S = \bigcap_{k=1}^{p} H_{\mathfrak{u}_k}$ *be a polyhedral cone in* $\mathbf{C}^n$.

The function $\mathfrak{g} : X\times\overline{X} \to \mathbf{C}^m$ *which defines the set* $Y = \{\mathbf{z} \in X / \mathfrak{g}(\mathbf{z}, \overline{\mathbf{z}}) \in S\}$, *is said to satisfy :*

1°. *the Arrow—Hurwicz—Uzawa* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0) \in Y\times\overline{Y}$ *if* $\mathfrak{g}$ *is differentiable at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ *and if*

$$(25)\quad \begin{cases} \operatorname{Re}\langle \overline{\nabla_{\mathbf{z}}\mathfrak{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\mathfrak{u}_k + \nabla_{\overline{\mathbf{z}}}\mathfrak{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\overline{\mathfrak{u}}_k}, \mathbf{z}\rangle > 0,\ k \in J \\ \operatorname{Re}\langle \overline{\nabla_{\mathbf{z}}\mathfrak{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\mathfrak{u}_k + \nabla_{\overline{\mathbf{z}}}\mathfrak{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\overline{\mathfrak{u}}_k}, \mathbf{z}\rangle \geqq 0,\ k \in I \end{cases}$$

has a solution $\mathbf{z} \in \mathbf{C}^n$, *where* I *and* J *are the sets defined by* (6).

2°. *the reverse concave* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0) \in Y\times\overline{Y}$ *if* $\mathfrak{g}$ *is differentiable at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ *and pseudo-convexe at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ *with respect to* $H_{\mathfrak{u}_k}$ *for all* $k \in E$ (E is the set defined by (6)).

3°. *the Kuhn—Tucker* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0) \in Y \times \overline{Y}$ [1] *if* $\mathbf{g}$ *is differentiable at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ *and for all* $\mathbf{z} \in \mathbf{C}^n$ *with*

$$(26) \qquad \overline{[\nabla_z \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T \mathbf{z} + [\nabla_{\bar z} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T \overline{\mathbf{z}}} \in \bigcap_{k \in E} H_{\mathbf{u}_k},$$

there exists an $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$ *and a function* $\mathbf{h}: [0, \varepsilon[\to \mathbf{C}^n$ *differentiable at* 0, *such that* $\mathbf{h}(0) = \mathbf{z}^0$, $\frac{d}{dt}\mathbf{h}(0) = \mathbf{z}$, *and* $\mathbf{h}(t) \in X$, $\mathbf{g}[\mathbf{h}(t), \overline{\mathbf{h}(t)}] \in S$ *for* $0 \leq t < \varepsilon$.

4°. *the weak* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0) \in Y \times \overline{Y}$ [6] *if* $\mathbf{g}$ *is differentiable at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ *and*

$$(27) \qquad \left.\begin{array}{l} \overline{\nabla_z \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\mathbf{v}} + \nabla_{\bar z}\mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\overline{\mathbf{v}} = \mathbf{0} \\ \mathrm{Re}\langle \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0), \mathbf{v}\rangle = 0 \\ \mathbf{v} \in S^* \end{array}\right\} \textit{imply } \mathbf{v} = \mathbf{0}.$$

LEMMA 5. *Let* X *be an open set in* $\mathbf{C}^n$, *let* $S = \bigcap_{k=1}^{p} H_{\mathbf{u}_k}$ *be a polyhedral cone in* $\mathbf{C}^m$, *let* $\mathbf{g}: X \times \overline{X} \to \mathbf{C}^m$ *and let* $\mathbf{z}^0 \in Y = \{\mathbf{z} \in X / \mathbf{g}(\mathbf{z}, \overline{\mathbf{z}}) \in S\}$.

(i) *If* $\mathbf{g}$ *satisfies the reverse concave* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$, *then* $\mathbf{g}$ *satisfies the Arrow—Hurwicz—Uzawa* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$.

(ii) *If* $\mathbf{g}$ *satisfies the reverse concave* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$, *then* $\mathbf{g}$ *satisfies the Kuhn—Tucker* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$.

(iii) *Let* $\mathbf{g}$ *be concave at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ *with respect to* S *and differentiable at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$. *If* $\mathbf{g}$ *satisfies Slater's* CCQ *with respect to* Y, *then* $\mathbf{g}$ *satisfies the weak* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$.

(iv) *Let* $\mathbf{g}$ *be concave at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ *with respect to* S *and differentiable at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$. *If* $\mathbf{g}$ *satisfies* Slater's CCQ, *or the strict* CCQ *with respect to* Y, *then* $\mathbf{g}$ *satisfies the Arrow—Hurwicz—Uzawa* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$.

(v) *Let* X *be convex, let* $\mathrm{int}\, S \neq \emptyset$, *let* $\mathbf{g}$ *be concave on* $X \times \overline{X}$ *with respect to* S *and differentiable at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$. *If* $\mathbf{g}$ *satisfies Karlin's* CCQ *with respect to* Y, *then* $\mathbf{g}$ *satisfies the Arrow—Hurwicz—Uzawa* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$.

(vi) *If* $\mathrm{int}\, S \neq \emptyset$, *then the weak* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ *implies the Arrow—Hurwicz—Uzawa* CCQ *at* $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$.

Proof. (i) Let g satisfy the reverse concave CCQ at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ and let E, I, J be defined by (6). Since $\mathbf{g}$ is pseudo-convex at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ with respect to $H_{\mathbf{u}_k}$ for all $k \in E$, we have that $J = \emptyset$. Then System (5) becomes

$$\mathrm{Re}\langle \overline{\nabla_z \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\mathbf{u}_k} + \nabla_{\bar z}\mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)\mathbf{u}_k, \mathbf{z}\rangle \geq 0, \ k \in I,$$

which has the solution $\mathbf{z} = \mathbf{0} \in \mathbf{C}^n$. Hence $\mathbf{g}$ satisfies the Arrow—Hurwicz—Uzawa CCQ at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$.

(ii) Let $\mathbf{g}$ satisfy the reverse CCQ at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ and let E, I, J be defined by (6). Let $\mathbf{z} \in \mathbf{C}^n$ satisfy

$$(28) \qquad \overline{[\nabla_z \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T \mathbf{z} + [\nabla_{\bar z} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T \overline{\mathbf{z}}} \in \bigcap_{k \in E} H_{\mathbf{u}_k}.$$

Define the function $\mathbf{h}(t) = \mathbf{z}^0 + t\mathbf{z}$, $t \in \mathbf{R}$. We have $\mathbf{h}(0) = \mathbf{z}^0$, $\frac{d}{dt}\mathbf{h}(0) = \mathbf{z}$. We will now show that there exists $\varepsilon > 0$ such that $\mathbf{h}(t) \in X$ and $\mathbf{g}[\mathbf{h}(t), \overline{\mathbf{h}(t)}] \in S$ for all $t \in [0, \varepsilon[$.

Since $\mathbf{z}^0 \in X$ and X is open there exists $\varepsilon_0 > 0$ such that

$$(29) \qquad \mathbf{h}(t) = \mathbf{z}^0 + t\mathbf{z} \in X \text{ for all } t \in [0, \varepsilon_0[.$$

From (28) it follows that

$$\overline{[\nabla_z \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T [\mathbf{h}(t) - \mathbf{z}^0] + [\nabla_{\bar z} \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T [\overline{\mathbf{h}(t)} - \overline{\mathbf{z}}^0]} \in \bigcap_{k \in E} H_{\mathbf{u}_k}$$

for all $t \in [0, \varepsilon_0[$. Since $\mathbf{g}$ is pseudo-convex at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$ with respect to $H_{\mathbf{u}_k}$ for all $k \in E$, we have

$$\mathbf{g}[\mathbf{h}(t), \overline{\mathbf{h}(t)}] - \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) \in H_{\mathbf{u}_k} \text{ for all } t \in [0, \varepsilon_0[\text{ and } k \in E,$$

hence

$$(30) \qquad \mathbf{g}[\mathbf{h}(t), \overline{\mathbf{h}(t)}] \in \bigcap_{k \in E} H_{\mathbf{u}_k} \text{ for all } t \in [0, \varepsilon_0],$$

because

$$\mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) \in \bigcap_{k \in E} H_{\mathbf{u}_k}.$$

Since $\mathbf{g}$ is differentiabe at $(\mathbf{z}^0, \overline{\mathbf{z}}^0)$, we have

$$\mathbf{g}[\mathbf{h}(t), \overline{\mathbf{h}(t)}] = \mathbf{g}(\mathbf{z}^0 + t\mathbf{z}, \overline{\mathbf{z}}^0 + t\overline{\mathbf{z}}) = \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0) + t\{[\nabla_z \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T\mathbf{z} + [\nabla_{\bar z}\mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T\overline{\mathbf{z}}\} + t\|\mathbf{z}\|\mathbf{a}(\mathbf{z}^0 + t\mathbf{z}, \mathbf{z}^0) \text{ for all } t \in [0, \varepsilon_0[,$$

hence

$$(31) \qquad \mathrm{Re}\langle \mathbf{g}[\mathbf{h}(t), \overline{\mathbf{h}(t)}], \mathbf{u}_k\rangle = \mathrm{Re}\langle \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0), \mathbf{u}_k\rangle + t\,\mathrm{Re}\langle [\nabla_z \mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T\mathbf{z} + [\nabla_{\bar z}\mathbf{g}(\mathbf{z}^0, \overline{\mathbf{z}}^0)]^T\overline{\mathbf{z}}, \mathbf{u}_k\rangle + t\|\mathbf{z}\|\,\mathrm{Re}\langle \mathbf{a}(\mathbf{z}^0 + t\mathbf{z}, \mathbf{z}^0), \mathbf{u}_k\rangle, \text{ for } t \in [0, \varepsilon_0[\text{ and } k \in \{1, \ldots, p\}$$

and

$$(32) \qquad \lim_{t \to 0} \mathrm{Re}\langle \mathbf{a}(\mathbf{z}^0 + t\mathbf{z}, \mathbf{z}^0), \mathbf{u}_k\rangle = 0 \text{ for all } k \in \{1, \ldots, p\}.$$

If $k \in L = \{k \in \{1, \dots, p\} / \operatorname{Re}\langle g(z^0, \bar z^0), u_k\rangle > 0\}$, then from (31) and (32) it follows that there exists $\varepsilon_k \in]0, \varepsilon_0[$ such that

(33) $$\operatorname{Re}\langle g[b(t), \overline{b(t)}], u_k\rangle < 0 \text{ for all } t \in [0, \varepsilon_k[, \ k \in L.$$

If we denote by $\varepsilon = \min\{\varepsilon_k / k \in L\}$ we have $\varepsilon > 0$ and from (33),

(34) $$g[b(t), \overline{b(t)}] \in \bigcap_{k \in L} H_{u_k} \text{ for all } t \in [0, \varepsilon[.$$

Since $S = (\bigcap_{k \in E} H_{u_k}) \cap (\bigcap_{k \in L} H_{u_k})$, from (30) and (34) we have

$$g[b(t), \overline{b(t)}] \in S \text{ for all } t \in [0, \varepsilon[,$$

and the Kuhn-Tucker CCQ at $(z^0, \bar z^0)$ is satisfied.

(iii) Let g satisfy Slater's CCQ with respect to Y, i.e. int $S \neq \emptyset$ and there exists $z^1 \in X$ such that $g(z^1, \bar z^1) \in$ int S, or equivalently,

(35) $$0 \neq v \in S^* \Rightarrow \operatorname{Re}\langle g(z^1, \bar z^1), v\rangle < 0.$$

If the weak CCQ at (z^0, z^0) is not satisfied, then there exists $v^0 \in S^*$, $v^0 \neq 0$ such that

(36) $$\overline{\nabla_z g(z^0, \bar z^0) v^0} + \nabla_{\bar z} g(z^0, \bar z^0) v^0 = 0$$
$$\operatorname{Re}\langle g(z^0, \bar z^0), v^0\rangle = 0.$$

The function g being concave at $(z^0, \bar z^0)$ with respect to S for all $v \in S^*$:

(37) $$\operatorname{Re}\langle g(z^1, \bar z^1), v\rangle \leqq \operatorname{Re}\langle g(z^0, \bar z^0), v\rangle +$$
$$+ \operatorname{Re}\langle \overline{\nabla_z g(z^0, \bar z^0) v} + \nabla_{\bar z} g(z^0, \bar z^0)\bar v, z^1 - z^0\rangle.$$

By letting $v = v^0 \in S^*$ in (37), from (36) we get that $\operatorname{Re}\langle g(z^1, \bar z^1), v^0\rangle \leqq 0$, which contradicts (35) for $v = v^0 \neq 0$, $v^0 \in S^*$.

(iv) By Lemma 4 (ii) the strict CCQ implies Slater's CCQ. If g satisfies Slater's CCQ with respect to Y, then int $S \neq \emptyset$ and there exists $z^1 \in X$ auch that $g(z^1, \bar z^1) \in$ int S.

Consider the sets E, I, J defined in (6). Since g is differentiable at $(z^0, \bar z^0)$ and concave at $(z^0, \bar z^0)$ with respect to S, we have

$$0 < \operatorname{Re}\langle g(z^1, \bar z^1), u_k\rangle \leqq \operatorname{Re}\langle g(z^0, \bar z^0), u_k\rangle +$$
$$+ \operatorname{Re}\langle [\nabla_z g(z^0, \bar z^0)]^T (z^1 - z^0) + [\nabla_{\bar z} g(z^0, \bar z^0)]^T (\bar z^1 - \bar z^0), u_k\rangle$$

for any $k \in \{1, \dots, p\}$, and hence

$$0 < \operatorname{Re}\langle \overline{\nabla_z g(z^0, \bar z^0) u_k} + \nabla_{\bar z} g(z^0, \bar z^0)\bar u_k, z^1 - z^0\rangle \text{ for any } k \in E.$$

System (5) has the solution $z = z^1 - z^0$ and the Arrow–Hurwicz–Uzawa CCQ at $(z^0, \bar z^0)$ is satisfied.

(v) Apply Lemma 4 (iii) and Lemma 5 (iv).

(vi) Let g satisfy the weak CCQ at $(z^0, \bar z^0)$, i.e. let (27) hold. If the Arrow–Hurwicz–Uzawa CCQ at $(z^0, \bar z^0)$ is not satisfied then System (25) has no solution $z \in C^n$ (E, I and J are again the sets defined in (6)). By Lemma 2, there exists $\mu = (\mu_k)_{k \in E} \geqslant 0$ such that

(38) $$\operatorname{Re}[\overline{\nabla_z g(z^0, \bar z^0) v} + \nabla_{\bar z} g(z^0, \bar z^0)\bar v]^H z \leqq 0 \text{ for all } z \in C^n,$$

where

$$v = \sum_{k=1}^{p} \lambda_k u_k \text{ and } \lambda_k = \begin{cases} \mu_k, & \text{if } k \in E \\ 0, & \text{if } k \in \{1, \dots, p\} \setminus E. \end{cases}$$

From (38) we have

(39) $$\overline{\nabla_z g(z^0, \bar z^0) v} + \nabla_{\bar z} g(z^0, \bar z^0)\bar v = 0.$$

Evidently,

(40) $$v \in S^* \text{ and } \operatorname{Re}\langle g(z^0, \bar z^0), v\rangle = 0.$$

If $v = 0$, then the system

$$\sum_{k \in E} \mu_k u_k = 0, \quad (\mu_k)_{k \in E} \geqslant 0,$$

has a solution, and by Lemma 1, the system

$$\operatorname{Re}\langle u_k, w\rangle > 0, \ k \in E,$$

has no solution $w \in C^m$, which contradicts int $S \neq \emptyset$. Consequently $v \neq 0$, but this together with (39) and (40) contradicts the fact that g satisfies the weak CCQ at $(z^0, \bar z^0)$.

THEOREM 3. *Let X ge an open set in C^n, let $S = \bigcap_{k=1}^{p} H_{u_k}$ be a polyhedral cone in C^m, let $A, B \in C^{m \times n}$ and $b \in C^m$, and let*

$$g(z, w) = Az + Bw + b \text{ for all } (z, w) \in X \times \bar X.$$

If $Y = \{z \in X / g(z, \bar z) \in S\}$ is nonempty, then g satisfies the reverse concave CCQ at any $(z^0, \bar z^0) \in Y \times \bar Y$.

Proof. Let $z^0 \in Y$. Evidently the function g is differentiable at $(z^0, \bar z^0)$ and $\nabla_z g(z^0, \bar z^0) = A^T$, $\nabla_{\bar z} g(z^0, \bar z^0) = B^T$. The function g being pseudoconcave at $(z^0, \bar z^0)$ with respect to H_{u_k} for all $k \in \{1, \dots, p\}$, it follows that the function g satisfies the reverse concave CCQ at $(z^0, \bar z^0)$.

COROLLARY 2. *Let* X, S, g *and* Y *be as in Theorem* 3. *Then the function* g *satisfies:*

a) *the Arrow–Hurwicz–Uzawa* CCQ *at any* $(z^0, \bar{z}^0) \in Y \times \bar{Y}$, *and*

b) *the Kuhn–Tucker* CCQ *at any* $(z^0, \bar{z}^0) \in Y \times \bar{Y}$.

Proof. In view of Theorem 3, the function g satisfies the reverse concave CCQ at any $(z^0, \bar{z}^0) \in Y \times \bar{Y}$. Now by applying Lemma 5 (i) and (ii), the corollary follows.

4. A Kuhn-Tucker Theorem in Complex Space

THEOREM 4. *Let* X *be a nonempty open set in* C^n, *let* $S = \bigcap_{k=1}^{p} H_{u_k}$ *be a polyhedral cone in* C^m *with nonempty interior, let* $f : X \times \bar{X} \to C$ *and* $g : X \times \bar{X} \to C^m$, *let* $Y = \{z \in X / g(z, \bar{z}) \in S\}$, *let* $z^0 \in Y$ *be a local minimum point of Problem* (P), *let* f *and* g *be differentiable functions at* $(z^0, \bar{z}^0)$ *and let* E *be the set defined by* (6).

Suppose in addition that one of following conditions holds:

(i) g *satisfies the Arrow–Hurwicz–Uzawa* CCQ *at* $(z^0, \bar{z}^0)$;

(ii) g *satisfies the Kuhn–Tucker* CCQ *at* $(z^0, \bar{z}^0)$;

(iii) g *satisfies the reverse concave* CCQ *at* $(z^0, \bar{z}^0)$;

(iv) g *satisfies the weak* CCQ *at* $(z^0, \bar{z}^0)$;

(v) g *satisfies Slater's* CCQ *with respect to* Y *and* g *is concave at* $(z^0, \bar{z}^0)$ *with respect to* S;

(vi) g *satisfies the strict* CCQ *with respect to* Y *and* g *is concave at* $(z^0, \bar{z}^0)$ *with respect to* S;

(vii) g *satisfies Karlin's* CCQ *with respect to* Y, X *is convex and* g *is concave on* $X \times \bar{X}$ *with respect to* S.

Then there exists $v \in \left(\bigcap_{k \in E} H_{u_k}\right)^* \subseteq S^*$ *such that*

$$\overline{\nabla_z f(z^0, \bar{z}^0)} + \nabla_{\bar{z}} f(z^0, \bar{z}^0) - \overline{\nabla_z g(z^0, \bar{z}^0)} - \nabla_{\bar{z}} g(z^0, \bar{z}^0)\bar{v} = 0 \tag{41}$$

$$\operatorname{Re}\langle g(z^0, \bar{z}^0), v\rangle = 0. \tag{42}$$

Proof. Let $z^0 \in Y$ be a local minimum point of Problem (P) and let E, I and J be the sets defined by (6).

In view of Lemma 5 we need to establish the theorem under the assumptions (i) and (ii).

(i) By Theorem 2 there exists a $\tau \in \mathbf{R}_+$ and $u \in \left(\bigcap_{k \in J} H_{u_k}\right)^* \subseteq S^*$ such that (I), (II) and (III) hold. We will show that J is empty. Then by Corollary 1 we have $\tau \neq 0$.

Assume J is nonempty. We will now show by contradiction that $\tau \neq 0$. Suppose that $\tau = 0$, then from (Ia) it follows that $u_J \neq 0$, and hence

$$(\lambda_k)_{k \in J} \geqslant 0. \tag{43}$$

Since g satisfies the Arrow–Hurwicz–Uzawa CCQ at $(z^0, \bar{z}^0)$, there exists a $z \in C^n$ such that

$$\begin{cases} \operatorname{Re}\langle \overline{\nabla_z g(z^0, \bar{z}^0)u_k} + \nabla_{\bar{z}} g(z^0, \bar{z}^0)\bar{u}_k, z\rangle > 0, & k \in J \\ \operatorname{Re}\langle \overline{\nabla_z g(z^0, \bar{z}^0)u_k} + \nabla_{\bar{z}} g(z^0, \bar{z}^0)\bar{u}_k, z\rangle \geqq 0, & k \in I. \end{cases} \tag{44}$$

From (I), (43) and (44) we have

$$\operatorname{Re}\langle \overline{\nabla_z g(z^0, \bar{z}^0)u} + \nabla_{\bar{z}} g(z^0, \bar{z}^0)\bar{u}, z\rangle =$$

$$= \operatorname{Re}\left[\sum_{k \in E} \lambda_k \langle \overline{\nabla_z g(z^0, \bar{z}^0)u_k} + \nabla_{\bar{z}} g(z^0, \bar{z}^0)\bar{u}_k, z\rangle\right] > 0,$$

which contradicts (II) for $\tau = 0$. Consequently $J = \emptyset$. Then by Corollary 1, it follows that $\tau > 0$. Dividing (II) and (III) by $\tau > 0$ and setting $v = (1/\tau)u \in \left(\bigcap_{k \in E} H_{u_k}\right)^* \subseteq S^*$, we get that (41) and (42).

(ii) Let $z \in C^n$ suct that (26) holds. Since g satisfies the Kuhn–Tucker CCQ at $(z^0, \bar{z}^0)$, there exists an $\varepsilon > 0$ and a function $b : [0, \varepsilon[\to C^n$ differentiable at 0, such that

$$b(0) = z^0, \qquad \frac{d}{dt} b(0) = z \tag{45}$$

and $b(t) \in X$, $g[b(t), \overline{b(t)}] \in S$ for all $t \in [0, \varepsilon[$.

Since z^0 is a local minimum of Problem (P) we have

$$\frac{d}{dt}\operatorname{Re}\{f[b(t), \overline{b(t)}]\}\big|_{t=0} \geqslant 0,$$

or equivalently,

$$\operatorname{Re}\left\{\frac{d}{dt} f[b(t), \overline{b(t)}]\right\}\Big|_{t=0} \geqslant 0. \tag{46}$$

From (45) and (46) it follows that

$$\mathrm{Re}\,\{[\nabla_{\mathbf{z}} f(\mathbf{z}^0, \bar{\mathbf{z}}^0)]^T\mathbf{z} + [\nabla_{\bar{\mathbf{z}}} f(\mathbf{z}^0, \bar{\mathbf{z}}^0)]^T\bar{\mathbf{z}}\} \geqslant 0.$$

Therefore the system

$$\begin{cases} \mathrm{Re}\,\langle \overline{\nabla_{\mathbf{z}} f(\mathbf{z}^0, \bar{\mathbf{z}}^0)} + \nabla_{\bar{\mathbf{z}}} f(\mathbf{z}^0, \bar{\mathbf{z}}^0), \mathbf{z}\rangle < 0, \\ \mathrm{Re}\,\langle \overline{\nabla_{\mathbf{z}} \mathbf{g}(\mathbf{z}^0, \bar{\mathbf{z}}^0)\mathbf{u}_k} + \nabla_{\bar{\mathbf{z}}} \mathbf{g}(\mathbf{z}^0, \bar{\mathbf{z}}^0)\bar{\mathbf{u}}_k, \mathbf{z}\rangle \geqslant 0, \quad k \in E, \end{cases}$$

has no solution $\mathbf{z} \in \mathbf{C}^n$. Then by Lemma 1 the system

$$\tau[\overline{\nabla_{\mathbf{z}} f(\mathbf{z}^0, \bar{\mathbf{z}}^0)} + \nabla_{\bar{\mathbf{z}}} f(\mathbf{z}^0, \bar{\mathbf{z}}^0)] - \\ -\sum_{k\in E} \mu_k [\overline{\nabla_{\mathbf{z}} \mathbf{g}(\mathbf{z}^0, \bar{\mathbf{z}}^0)\mathbf{u}_k} + \nabla_{\bar{\mathbf{z}}} \mathbf{g}(\mathbf{z}^0, \bar{\mathbf{z}}^0)\bar{\mathbf{u}}_k] = 0, \tag{47}$$

has a solution $(\tau, \mu_E) \geqslant \mathbf{0}$ with $\tau \geqslant 0$, where $\mu_E = (\mu_k)_{k\in E}$. Since $\tau \geqslant 0$ is equivalent to $\tau > 0$, from (47) it follows that

$$\overline{\nabla_{\mathbf{z}} f(\mathbf{z}^0, \bar{\mathbf{z}}^0)} + \nabla_{\bar{\mathbf{z}}} f(\mathbf{z}^0, \bar{\mathbf{z}}^0) - \overline{\nabla_{\mathbf{z}} g(\mathbf{z}^0, \bar{\mathbf{z}}^0)\mathbf{v}} - \nabla_{\bar{\mathbf{z}}} \mathbf{g}(\mathbf{z}^0, \bar{\mathbf{z}}_0)\bar{\mathbf{v}} = \mathbf{0},$$

where $\mathbf{v} = \sum_{k=1} \frac{\mu_k}{\tau} \mathbf{u}_k$.

Let us denote by

$$\lambda_k = \begin{cases} \mu_k/\tau, & k \in E \\ 0, & k \in \{1, \ldots, p\}/E. \end{cases}$$

Then $\mathbf{v} = \sum_{k=1}^{p} \lambda_k \mathbf{u}_k$ and $\mathbf{v} \in \left(\bigcap_{k\in E} H_{\mathbf{u}_k}\right)^* \subseteq S^*$. Moreover $\mathrm{Re}\,\langle \mathbf{g}(\mathbf{z}^0, \bar{\mathbf{z}}^0), \mathbf{v}\rangle = 0$.

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