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DISTANCES FOR VECTOR-VALUED NORMS

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1. Introduction. Let E be a lienar space over the (real or complex) field **K** endomed with a (vector-valued) norm $\|\cdot\|$ with values in the plane $\mathbf{R}^2$ (with its natural partial ordering), i.e., $\|\cdot\|: E \to \mathbf{R}^2$ satisfies the following axioms: 1°) $\|x\| > 0$ if $x \neq 0$ and $\|0\| = 0$; 2°) the elements $\|x_1 + x_2\|$, $\|x_1\| + \|x_2\|$ are comparable and $\|x_1+x_2\| \leqslant \|x_1\|+\|x_2\|$ $(x_1, x_2 \in E)$; 3°) $\|\lambda x\| = |\lambda| \|x\|$ $(\lambda \in \mathbf{K},\ x \in E)$.

When G is a nonempty set of E and $x \in E$, the „distance" of x to G for the case of norms with values in $\mathbf{R}^2$, denoted by DIST (x, G), was introduced in [1] using the notion of „infimum" of any set $A \subset \mathbf{R}^2$, which is no longer a point, but a set in the plane $\mathbf{R}^2$, denoted by INF A (see Definition 1 below). Consequentely DIST(x, G) = INF $\{\|x - g\| : g \in G\}$ is in general a subset pf $\mathbf{R}^2$. In [3] we extended some well-known properties of the numbers dist (.,.) (the distance of a point to a subset of a normed linear space) for the subsets of the plane $\mathbf{R}^2$, DIST (.,.). We have introduced a suitable partial pre-order relation $\lesssim$ for subsets of $\mathbf{R}^2$ which are bounded from below in $\mathbf{R}^2$, in such a way that if we have an inequality where the numbers dist (.,.) appear, then the relation remain valid if we replace dist with DIST and $\leqslant$ with $\lesssim$.

Another definition of the „infimum" of a set $A \subset \mathbf{R}^2$ was also introduced in [1] (see also [2], §4) denoted by INF$_1$ A which is in general larger than INF A. Consequently, we can define a „new" distance of x to G, denoted DIST$_1$ (x, G) for the case of norms with values in $\mathbf{R}^2$ ([1], [2]). In the present paper we shall study this new distance following the same line as in [3]. We shall introduce another partial pre-order relation $\lesssim_1$ for subsets of $\mathbf{R}^2$ which are bounded from below, so that the properties of DIST$_1$ (.,.) extend again the properties of dist (.,.). Some results will be similar with those of [3]. When the proofs will be similar with the ones of [3], we shall omit them. We conclude the paper showing how the results of [3] can be extended for the case of vector-valued norms with values in a reflexive Banach lattice.

2. We recall the definitions of INF A and INF_1 A for a subset $A \subset \mathbf{R}^2$ ([1], [2]). For a subset $A \subset \mathbf{R}^2$ we denote by $\bar{A}$ the closure of A in $\mathbf{R}^2$.

Definition 1 ([1], [2]). *Let $A \subset \mathbf{R}^2$ and let $p \in \mathbf{R}^2$. We shall say that $p \in \text{INF } A$ if the following two conditions are satisfied :*

1°) *There exists no $a \in A$ such that $a < p$.*

2°) $p \in \bar{A}$.

This definition is a particular case of the definition of weak or strong extremum with respect to a closed convex cone, of a set A in a Banach space ([4], [5], see also Definiton 6 below).

Definition 2 ([1], [2], *Let $A \subset \mathbf{R}^2$ and let $p \in \mathbf{R}^2$. We shall say that $p \in \text{INF}_1\, A$ if the following two conditions are satisfied :*

1°) *There exists no $a \in A$ such that $a < p$.*

2°) *For each real $\varepsilon > 0$ there exists an element $a \in A$ such that $a \leqslant p + \varepsilon e$* (where $e = (1, 1) \in \mathbf{R}^2$).

As it was remarked in [2], we have always :

$$\text{INF } A = \bar{A} \cap \text{INF}_1\, A \tag{1}$$

and Proposition 1, Remark 1a and Remark 1b of [2] are also true for INF_1 (see Lemmas 1—3 below).

Let $\mathcal{M}$ be the collection of all nonempty subsets of $\mathbf{R}^2$ which are bounded from below (i.e., there exists $r \in \mathbf{R}^2$, which depends on A, such that $r \leqslant a$ for all $a \in A$).

LEMMA 1 ([2]). *Let $A \in \mathcal{M}$ and $a \in A$. Then there exists $p \in \text{INF}_1\, A$ such that $p \leqslant a$. In particular, $\text{INF}_1\, A \neq \emptyset$.*

LEMMA 2 ([2]). *For each nonempty subset $A \subset \mathbf{R}^2$ we have $\text{INF}_1\, \bar{A} \subset \text{INF}_1\, A$.*

For $A \in \mathcal{M}$, let $\mu_i = \inf \{\alpha_i : (\alpha_1, \alpha_2) \in A\}$, $i = 1, 2$.

LEMMA 3 ([2]). *Let $A \in \mathcal{M}$. If $m = (\mu_1, \mu_2) \in A$, then $\text{INF}_1\, A = \{m\}$.*

Remark 1. Let $A \in \mathcal{M}$ and let $p \in \text{INF}_1\, A$. Then there exists $\bar{a} \in \text{INF } A$ $(\subset \bar{A})$ such that $\bar{a} \leqslant p$. Indeed, since $p \in \text{INF}_1\, A$, for each positive integer n there exists $a_n \in A$ such that $a_n \leqslant p + e/n$. Hence, since $A \in \mathcal{M}$, the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded in $\mathbf{R}^2$. Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a convergent subsequence of $\{a_n\}_{n=1}^{\infty}$ and let $\bar{a} = \lim_{k\to\infty} a_{n_k}$. Then $\bar{a} \in \bar{A}$ and clearly we have $\bar{a} \leqslant p$. Since $p \in \text{INF}_1\, A$, it follows $\bar{a} \in \text{INF } A$.

Remark 2. For each $A \in \mathcal{M}$ we have $\text{INF}_1\, \bar{A} = \text{INF } \bar{A}$. Indeed, by (1) we have $\text{INF } \bar{A} \subset \text{INF}_1\, \bar{A}$. Let now $p \in \text{INF}_1\, \bar{A}$. By Remark 1, there exists $\bar{a} \in \text{INF } \bar{A}$ $(\subset \bar{A})$ such that $\bar{a} \leqslant p$. Since $p \in \text{INF}_1\, \bar{A}$, it follows $p = \bar{a} \in \text{INF } \bar{A}$.

PROPOSITION 1. *Let $A, B \in \mathcal{M}$ and $\lambda \in \mathbf{R}$, $\lambda \geqslant 0$. Then :*

i) $\text{INF}_1\, (\text{INF}_1\, A) \subset \text{INF}_1\, A$

ii) $\text{INF}_1\, (\text{INF}_1\, \bar{A}) = \text{INF}_1\, \bar{A} = \text{INF}_1\, (\text{INF}_1\, A)$

iii) $\text{INF}_1\, (A \cup B) \subset (\text{INF}_1\, A) \cup (\text{INF}_1\, B)$

iv) $\text{INF}_1\, (\bar{A} + \bar{B}) \subset (\text{INF}_1\, \bar{A}) + (\text{INF}_1\, \bar{B})$

v) $\text{INF}_1\, (\lambda A) = \lambda\, \text{INF}_1\, A$.

Proof. Some easy modifications in the proof of Proposition 1 of [3], using Lemmas 1, 2, Remarks 1, 2 and formula (1) above show i) — v). Use also Proposition 1 of [3] for the proofs of ii) and iv). For example iv) is an immediate consequence of Remark 2 above and [3] (Remark 1 and Proposition 1 (iv)).

In [3] we have introduced the following partial pre-order relation on $\mathcal{M}$:

Definition 3 ([3]). *For $A, B \in \mathcal{M}$, we define $A \lesssim B$ if $\text{INF } (\bar{A} \cup \bar{B}) = \text{INF } A$.*

Now, we shall introduce another partial pre-order relation on $\mathcal{M}$.

Definition 4. *For $A, B \in \mathcal{M}$, we define $A \lesssim_1 B$ if $\text{INF}_1\, (A \cup B) = \text{INF}_1\, A$.*

Remark 3. If $A, B \in \mathcal{M}$, $B \subset A$, then $A \lesssim_1 B$.

LEMMA 4. *Let $A, B \in \mathcal{M}$. The following two assertions are equivalent :*

i) $A \lesssim_1 B$

ii) *Fir each $b \in B$ and each $q \in \mathbf{R}^2$ such that $b < q$, there exists $a \in A$, $a < q$.*

Proof. i) $\Rightarrow$ ii). Suppose we have i) and let $b \in B$ and $q \in \mathbf{R}^2$ such that $b < q$. Then $q \notin \text{INF}_1\, (A \cup B)$ and by i), $q \notin \text{INF}_1\, A$. By Lemma 1, there exists $p \in \text{INF}_1\, (A \cup B)$ such that $p \leqslant b < q$. By i), $p \in \text{INF}_1\, A$ and so, for each $\varepsilon > 0$ there exists $a \in A$, $a \leqslant p + \varepsilon e < q + \varepsilon e$. Therefore q satisfies condition 2°) of Definition 2. Since $q \notin \text{INF}_1\, A$, by Definition 2 there exists $a \in A$, $a < q$.

ii) $\Rightarrow$ i). Suppose we have ii) and we must show that :

$$\text{INF}_1\, (A \cup B) = \text{INF}_1\, A \tag{2}$$

Let $p \in \text{INF}_1\, (A \cup B)$. Then clearly there exists no $a \in A$ with $a < p$. Let $\varepsilon > 0$. Since $p \in \text{INF}_1\, (A \cup B)$, there exists $b \in A \cup B$ such that $b \leqslant p + (\varepsilon/2)e$. If $b \in A$ then $b \leqslant p + \varepsilon e$. If $b \in B$, then $b \leqslant p + \varepsilon e$ whence by ii) there exists $a \in A$ with $a < p + \varepsilon e$. So, we have proved the inclusion $\subset$ in (2). Let now $p \in \text{INF}_1\, A$ and $b \in A \cup B$ with $b < p$. Then

$b \in B$ and by ii) there exists $a \in A$ such that $a < p$, contradicting $p \in \mathrm{INF}_1 A$. Thus p satisfies condition 1°) of Definition 2 for the set $A \cup B$, and obviously p satisfies condition 2°) for $A \cup B$ too (since $p \in \mathrm{INF}_1 A$). Therefore $p \in \mathrm{INF}_1 (A \cup B)$, which completes the proof.

Remark 4. Let $A, B \in \mathcal{M}$. It is an immediate consequence of Lemma 4, that if for each $b \in B$ there exists $a \in A$ with $a \leq b$, then $A \lesssim_1 B$. The converse is not always true as simple examples show.

PROPOSITION 2. *The relation* $\lesssim_1$ *is a partial pre-order relation on* $\mathcal{M}$.

Proof. By Definition 2, $\lesssim_1$ is reflexive. To show that $\lesssim_1$ is transitive, use Lemma 4.

PROPOSITION 3. *Let* $A, B \in \mathcal{M}$. *If* $A \lesssim_1 B$ *then we have* $A \lesssim B$.

Proof. Let $b \in B$. By Lemma 1, there exists $p \in \mathrm{INF}_1 (A \cup B)$ such that $p \leq b$. Since $A \lesssim_1 B$, it follows $p \in \mathrm{INF}_1 A$. By Remark 1, there exists $\bar{a} \in \bar{A}$ with $\bar{a} \leq p$. Therefore $\bar{a} \leq b$ and by [3], Lemma 1 iii) $\Rightarrow$ i) we get $A \lesssim B$, which completes the proof.

Simple examples show that the converse of Proposition 3 is not always true.

Remark 5. One can show that for $A, B \in \mathcal{M}$, the condition $A \lesssim_1 B$ is equivalent with the following two conditions: i) $A \lesssim B$; ii) For each $p \in \mathrm{INF}_1 A$ there exists no $b \in B$ with $b < p$.

THEOREM 1. *Let* $A, B \in \mathcal{M}$. *The following assertions are equivalent:*

i) $A \lesssim_1 B$

ii) $A \cup C \lesssim_1 B \cup C$ *for each* $C \in \mathcal{M}$

iii) $A + C \lesssim_1 B + C$ *for each* $C \in \mathcal{M}$

iv) $\lambda A \lesssim_1 \lambda B$ *for each* $\lambda \in \mathbf{R}, \lambda \geq 0$

Each of the above assertions implies:

v) $\mathrm{INF}_1 A \lesssim_1 \mathrm{INF}_1 B$

Proof. The proof of the equivalences i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv) is similar with the proof of [3], Theorem 1 (using Lemma 4 above) and we omit it.

i) $\Rightarrow$ v). Let $p \in \mathrm{INF}_1 B$. By Remark 1, there exists $\bar{b} \in \bar{B}$ with $\bar{b} \leq p$. By i) and Proposition 3, we get $A \lesssim B$, whence by [3] (Lemma 1 i) $\Rightarrow$ ii)) there exists $\bar{a} \in \bar{A}$ such that $\bar{a} \leq \bar{b}$ $(\leq p)$. By Lemma 1 there exists $q \in \mathrm{INF}_1 \bar{A}$ with $q \leq \bar{a}$. By Lemma 2, $q \in \mathrm{INF}_1 A$ and since $q \leq p$, by Remark 4 we obtain v), which completes the proof.

In [3], Theorem 1 we have shown that for $A, B \in \mathcal{M}$, the relation $A \lesssim B$ is equivalent with $\mathrm{INF}\, A \lesssim \mathrm{INF}\, B$. Simple examples show that the relation $\mathrm{INF}_1 A \lesssim_1 \mathrm{INF}_1 B$ does not imply $A \lesssim_1 B$.

Some immediate consequences of Theorem 1 are:

COROLLARY 1. *Let* $A, B, C, D \in \mathcal{M}$ *be such that* $A \lesssim_1 B$ *and* $C \lesssim_1 D$. *Then:*

i) $A \cup C \lesssim_1 B \cup D$

ii) $A + C \lesssim_1 B + D$

Corollary 2. *Let* $A, B \in \mathcal{M}$ *be such that* $B \subset A$. *Then* $\mathrm{INF}_1 A \sim_1 \mathrm{INF}_1 B$.

Notation. For $A, B \in \mathcal{M}$ we shall use the notation $A \sim_1 B$ if both $A \lesssim_1 B$ and $B \lesssim_1 A$ hold.

Remark 6. For $A, B \in \mathcal{M}$ we have $A \sim_1 B$ if and only if $\mathrm{INF}_1 A = \mathrm{INF}_1 B$. Indeed, if $A \sim_1 B$ then $\mathrm{INF}_1 A = \mathrm{INF}_1 (A \cup B) = \mathrm{INF}_1 B$. Conversely, suppose $\mathrm{INF}_1 A = \mathrm{INF}_1 B$. We show that $A \lesssim_1 B$ the proof for $B \lesssim_1 A$ being similar. Let $b \in B$ and $q \in \mathbf{R}^2$ with $b < q$. Then $q \notin \mathrm{INF}_1 B$ and by hypothesis, $q \notin \mathrm{INF}_1 A$. By Lemma 1, there exists $p \in \mathrm{INF}_1 B$ such that $p \leq b$. Then $p \in \mathrm{INF}_1 A$ and so for each $\varepsilon > 0$ there existes $a \in A$ with $a \leq p + \varepsilon e \leq b + \varepsilon e < q + \varepsilon e$. Since q satisfies condition 2°) of Definition 2 and $q \notin \mathrm{INF}_1 A$, it follows that there is $a \in A$ with $a < q$. By Lemma 4 we get $A \lesssim_1 B$.

PROPOSITION 4. *For each* $A \in \mathcal{M}$ *we have* $\mathrm{INF}_1 A \sim_1 \mathrm{INF}_1 \bar{A}$.

Proof. Use Remark 6 and Proposition 1 ii).

In [3] for $A, B \in \mathcal{M}$ we used the notation $A \sim B$ if both $A \lesssim_1 B$ and $B \lesssim_1 A$ hold. In Proposition 3 of [3] we have shown that for each $A \in \mathcal{M}$ we have $A \sim \mathrm{INF}\, A \sim \mathrm{INF}\, \bar{A}$. Simple examples show that the relation $A \sim_1 \mathrm{INF}_1 A$ is not always true.

PROPOSITION 5. *Let* $A, B \in \mathcal{M}$. *We have:*

i) $\mathrm{INF}_1 (A \cup B) \sim_1 (\mathrm{INF}_1 A) \cup (\mathrm{INF}_1 B)$

ii) $\mathrm{INF}_1 (A + B) \sim_1 (\mathrm{INF}_1 A) + (\mathrm{INF}_1 B)$

Proof. i) By Corollary 2 we obtain $\mathrm{INF}_1 (A \cup B) \lesssim_1 \mathrm{INF}_1 A$ and $\mathrm{INF}_1 (A \cup B) \lesssim_1 \mathrm{INF}_1 B$, whence by Corollary 1 i) we get $\mathrm{INF}_1 (A \cup B) \lesssim_1 (\mathrm{INF}_1 A) \cup (\mathrm{INF}_1 B)$. Let now $p \in \mathrm{INF}_1 (A \cup B)$. By Proposition 1 iii), it follows $p \in (\mathrm{INF}_1 A) \cup (\mathrm{INF}_1 B)$, whence by Remark 4 $(\mathrm{INF}_1 A) \cup (\mathrm{INF}_1 B) \lesssim_1 \mathrm{INF}_1 (A \cup B)$, which shows i).

ii) Let $p \in (\mathrm{INF}_1 A) + (\mathrm{INF}_1 B)$. Then $p = p_1 + p_2$ for some $p_1 \in \mathrm{INF}_1 A$ and $p_2 \in \mathrm{INF}_1 B$. By Remark 1, there is $\bar{a} \in \bar{A}$ with $\bar{a} \leq p_1$ and $\bar{b} \in \bar{B}$ with $\bar{b} < p_2$. By Lemma 1, there exists $q \in \mathrm{INF}_1 (\bar{A} + \bar{B})$ such that $q \leq \bar{a} + \bar{b}$. By Lemma 2 and [3], Remark 1 it follows $q \in \mathrm{INF}_1 (A + B)$. Since $q \leq \bar{a} + \bar{b} \leq p_1 + p_2 = p$, by Remark 4 we obtain $\mathrm{INF}_1 (A + B) \lesssim_1 (\mathrm{INF}_1 A) + (\mathrm{INF}_1 B)$. Let now $p \in \mathrm{INF}_1 (A + B)$. By Remark 1, there exists $d \in \overline{A + B}$ $(= \bar{A} + \bar{B})$ with $d \leq p$. Then $d = \bar{a} + \bar{b}$ for some $\bar{a} \in \bar{A}$ and $\bar{b} \in \bar{B}$. By Lemma 1, there is $p_1 \in \mathrm{INF}_1 \bar{A}$ and $p_2 \in \mathrm{INF}_1 \bar{B}$ with $p_1 \leq \bar{a}$, $p_2 \leq \bar{b}$. By Lemma 2 we get $p_1 \in \mathrm{INF}_1 A$, $p_2 \in \mathrm{INF}_1 B$. Hence, $p_1 + p_2 \in (\mathrm{INF}_1 A) + (\mathrm{INF}_1 B)$ and $p_1 + p_2 \leq \bar{a} + \bar{b} = d \leq p$. By

Remark 4 we obtain $(\text{INF}_1 A)+(\text{INF}_1 B) \lesssim_1 \text{INF}_1 (A+B)$, which completes the proof.

In [3], Proposition 4 we proved for $\sim$ and INF a stronger result then Proposition 5 above. Namely, we have for each $A, B \in \mathcal{M}$ $A \cup B \sim (\text{INF } A) \cup (\text{INF } B)$ and $A + B \sim (\text{INF } A) + (\text{INF } B)$. These are not always true for $\sim_1$ and INF_1 as simple examples show.

3. Let E be a linear space endowed with a vector-valued norm with values in $\mathbf{R}^2$ and let G be a nonempty subset of E. We shall denote by $\overline{G}$ the closure of G in the norm $\|\cdot\|$. For $x \in E$ let $\text{DIST}_1(x, G)$ be defined by:

$$\text{DIST}_1(x, G) = \text{INF}_1\{\|x - g\| : g \in G\}$$

Clearly, $\text{DIST}_1(x, G) \subset \{p \in \mathbf{R}^2 : p \geqslant 0\}$, whence $\text{DIST}_1(x, G) \in \mathcal{M}$. To extend some known results of dist (., .) for DIST_1 (., .) we need the following extension of the notion of convex function (see also [3], for another definition useful for DIST (., .)). Let E be a linear space and let $U : E \to 2^{\mathbf{R}^2}$ be a set-valued mapping, where $2^{\mathbf{R}^2}$ means the set of all nonempty subsets of $\mathbf{R}^2$.

Definition 5. *The set-valued mapping $U : E \to 2^{\mathbf{R}^2}$ is called convex$_1$ if the following two conditions are satisfied:*

1°) *$U(x) \in \mathcal{M}$ for each $x \in E$.*

2°) *$U(\lambda x_1 + (1 - \lambda)x_2) \lesssim_1 \lambda U(x_1) + (1 - \lambda)U(x_2)$ for each $x_i \in E$, $i = 1, 2$ and each $\lambda \in \mathbf{R}$ with $0 \leqslant \lambda \leqslant 1$.*

The following theorem extends for DIST_1 (., .) some properties of the usual distance dist (., .) (see e.g., [6], Theorem 6.5 for the corresponding results for dist (.|, .); see also [3], Theorem 2 for the results concerning DIST (., .)). We shall denote by p either the element p or the set $\{p\}$.

THEOREM 2. *Let E be a linear space endowed with a norm with values in $\mathbf{R}^2$, G a nonempty subset of E and $x, y \in E$. We have:*

1°) i) $0 \lesssim_1 \text{DIST}_1(x, G)$

ii) $\text{DIST}_1(x, G) = 0$ *for each* $g \in G$

iii) $\text{DIST}_1(x, G) \lesssim_1 \text{DIST}(y, G) + \|x - y\|$

iv) $\text{DIST}_1(x, G) \lesssim_1 \|x - g\|$ *for each* $g \in G$

v) $\text{DIST}_1(x, G) \sim_1 \text{DIST}_1(x, \overline{G})$

2°) If $G_1 \subset G$, $G_1 \neq \emptyset$, *then* $\text{DIST}_1(x, G) \lesssim_1 \text{DIST}_1(x, G_1)$.

3°) *Let G be a convex subset of E. Then the set-valued mapping $\text{DIST}_1(., G) : E \to 2^{\mathbf{R}^2}$ is convex$_1$.*

4°) *If G is a linear subspace of E and $\gamma \in \mathbf{K}$, then:*

i) $\text{DIST}_1(\gamma x, G) = |\gamma| \text{DIST}_1(x, G)$

ii) $\text{DIST}_1(x + y, G) \lesssim_1 \text{DIST}_1(x, G) + \text{DIST}_1(y, G)$

Proof. Easy modifications in the proof of [3], Theorem 2 show the above result. We only want to note that though in [3] we have used some results which are not valid for INF_1, these can be avoided using the results of this paper.

The next result extends the continuity property of dist (., G) for DIST_1 (., G). However, it says no more than Proposition 5 of [3] as one can see by the proof below.

PROPOSITION 6. *Let E be a linear space endowed with a norm with values in $\mathbf{R}^2$, $\emptyset \neq G \subset E$ and let $\{x_n\}_{n=0}^{\infty} \subset E$ be such that $\lim_{n\to\infty} \|x_n - x_0\| = 0$. Then for each $p_0 \in \text{INF}_1 \text{DIST}_1(x_0, G)$, there exists $p_n \in \text{INF}_1 \text{DIST}_1(x_n, G)$, $n = 1, 2, \ldots$ such that $\lim_{n\to\infty} p_n = p_0$.*

Proof. By Proposition 1 ii), Remark 2 and [3], Proposition 1 ii) we obtain $\text{INF}_1 \text{DIST}_1(x, G) = \text{INF DIST}(x, G)$ for each $x \in E$, whence the result follows by Proposition 5 of [3].

Proposition 6 of [3] is stronger than the same result where we replace DIST with DIST_1 (this being a consequence of the fact that $\text{DIST}(x, G) \subset \text{DIST}_1(x, G)$, for each $x \in E$), and we omit it.

The results of section 2 and 3 remain valid if we replace everywhere $\mathbf{R}^2$ by $\mathbf{R}^n$, the generalizations being straightforward.

4. For a subset A of a normed linear space X, we shall denote by w–cl A, the closure of A for the $\sigma(X, X^*)$-topology. Let K be a closed convex cone of X (with vertex at the origin). Then K induces a partial pre-order relation on X, denoted by $\leqslant$ and defined by $x \leqslant y$ if $y - x \in K$.

Definition 6 ([4], [5]). *Let A be a subset of the normed linear space X and K a closed convex cone of X. The element $p \in X$ is called a weak extremum of A with respect to K if the following two conditions are satisfied:*

1°) *There exists no $a \in A$ such that $a < p$.*

2°) *$p \in$ w–cl A.*

In the sequel X will be a reflexive Banach lattice, $K = \{x \in X : x \geqslant 0\}$ and for each nonempty set $A \subset X$ we denote by INF A the set of all weak extrema of A with respect to K. Note that for $X = \mathbf{R}^2$, the set of all weak extrema with respect to K is nothing else than INF A given by Definition 1.

Let $\mathcal{M}$ be the set of all nonempty subsets of X which are bounded from below.

Using Zorn's Lemma one can show the following generalization of Proposition 1 of [2]. The last statement of the next lemma was announced in [5], p. 141, statement (ii), but the hypotheses on X are different.

LEMMA 5. *Let X be a reflexive Banach lattice, $A \in \mathcal{M}$ and $a \in A$. Then there exists $p \in$ INF A such that $p \leqslant a$. In particular INF $A \neq \emptyset$.*

All results of [3], §2 are true if we replace everywhere $\mathbf{R}^2$ with a reflexive Banach lattice X, and the closure of any set A with w$-$cl A, the proofs being similar, (Note that in [3] Definition 1 we replace also $\bar{A}$ and $\bar{B}$ with w$-$cl A and w$-$cl B).

If E is a linear space endowed with a norm $\|\cdot\|$ with values in K, G a nonempty subset of E and $x \in E$ then for DIST $(x, G) =$ INF $\{\|x - g\| : g \in G\} \subset K$, the results of [3], § 3 are also true, the proofs being similar with those given in [3].

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