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ON THE HIERARCHY OF THE EFFICIENT POINTS IN LINEAR MULTIPLE OBJECTIVE PROGRAMS WITH ZERO-ONE VARIABLES

by

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Multiple objective programs with continuous variables have been exhaustively treated in the literature [3], [4], [7], [9] — [11]. However very little has been done for the zero-one case. In [1] and [2], theoretical results for zero-one linear multiple objective programs are developed and two algorithms to determine all efficient solutions are obtained.

There are several problems which arise in multiple objective programs with zero-one variables. First we need to determine an efficient (nondominated) solution, if it exists, or to establish that the set of all efficient solutions is empty. Then using a certain process of enumerating the efficient solutions it is always possible to construct the whole efficient set of the given multiple objective programs with zero-one variables. Once the set of efficient solutions has been constructed, we often need to optimize another criterion (a supercriterion) on the efficient set.

Sometimes in economy besides the optimal solution of the supercriterion on the efficient set, a number of other efficient solutions in the neighbourhood of the optimal solution are useful. So, the problem which arises is to list the elements of the efficient set in the from $x^1, \ldots, x^r$ such that if F is a supercriterion function, then $F(x^k) \geqslant F(x^{k+1})$ for each $k \in \{1, \ldots, r-1\}$.

Let $\mathbf{A} = (a_{ij})_{j=1,\ldots,n}^{i=1,\ldots,m}$ be an $m \times n$ matrix and let $\mathbf{x} = (x_1, \ldots x_n)^T \in \mathbf{R}^m$, $\mathbf{c}_h = (c_{h1}, \ldots, c_{hn})^T \in \mathbf{R}^n$, $h \in \{1, \ldots, s\}$ and $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbf{R}^m$. For future refernce we define the following sets:

$$I = \{1, \ldots, m\},\ J = \{1, \ldots, n\},\ H = \{1, \ldots, s\},$$
$$B = \{0, 1\},\ B^n = \underbrace{Bx \ldots xB}_{n},\ E \in \{B^n \mid \mathbf{Ax} \leqslant \mathbf{b}\}.$$

Here $\leqslant$ denotes the coordinatewise ordering. Further, we need also the ordering $\prec$ which is defined as follows: $\mathbf{x} \prec \mathbf{y}$ if $x_j \leqslant y_j$, $j \in J$ with at least one strict inequality.

By (MP) we denote the multiple objective program which possesses the constraint set E and the objective functions f_h, $h \in H$, where $f_h : R^n \to \mathbf{R}$ and

$$f_h(\mathbf{x}) = \mathbf{c}_h^T x \quad \text{for all } \mathbf{x} \in \mathbf{R}^n.$$

Let $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^s$, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_s(\mathbf{x}))^T$ for all $\mathbf{x} \in \mathbf{R}^n$.

D e f i n i t i o n 1. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are points belonging to* E, *we say that* $\mathbf{x}$ *is upper dominated by* $\mathbf{y}$ *if* $\mathbf{f}(\mathbf{x}) \prec \mathbf{f}(\mathbf{y})$.

The set of all points of E dominated by an element $\mathbf{y} \in E$ is denoted by $X(E, \mathbf{y})$.

In order to illustrate definition 1, we consider the following example:

$$\text{let } \mathbf{A} = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \mathbf{b} = (1, 1)^T \text{ and } \mathbf{f} : \mathbf{R}^3 \to \mathbf{R}^2,$$

$$f(x_1, x_2, x_3) = (2x_1 + 3x_2 + 6x_3,\ x_2 - x_3)^T \quad \text{for all } \mathbf{x} \in \mathbf{R}^3.$$

The set E is $E = \{(0, 0, 0)^T, (0, 0, 1)^T, (1, 0, 0)^T, (1, 1, 0)^T\}$. We have $\mathbf{f}(0, 0, 0,) = (0, 0)^T$, $\mathbf{f}(0, 0, 1) = (6, -1)^T$, $\mathbf{f}(1, 0, 0) = (2, 0)^T$, $\mathbf{f}(1, 1, 0) = (5, 1)^T$.
The point $(0, 0, 0)^T$ is upper dominated by the points $(1, 1, 0)^T$ and $(1, 0, 0)^T$. The point $(1, 0, 0)^T$ is upper dominated by the point $(1, 1, 0)^T$. The points $(1, 1, 0)^T$ and $(0, 0, 1)^T$ are not upper dominated. It is easy to see that $X(E, (0, 0, 0)^T) = \emptyset$, $X(E, (0, 0, 1)^T) = \emptyset$, $X(E, (1, 0, 0)^T) = \{(0, 0, 0)^T\}$, $X(E, (1, 1, 0)^T) = \{(0, 0, 0)^T, (1, 0, 0)^T\}$.

D e f i n i t i o n 2. *A point* $\mathbf{x} \in E$ *is said to be an efficient solution to* (MP) *if there is no* $\mathbf{y} \in E$ *such that* $\mathbf{f}(\mathbf{x}) \prec \mathbf{f}(\mathbf{y})$.

The set of all efficient solutions to (MP) is denoted by EF and the set of all non-efficient solutions to (MP) by CEF.

In the previous example the points $(1, 1, 0)^T$ and $(0, 0, 1)^T$ are efficient solutions for (MP). Therefore $EF = \{(0, 0, 1)^T, (1, 1, 0)^T\}$ and $CEF = \{(1, 0, 0)^T, (0, 0, 0)^T\}$.

R e m a r k 1. *If there is an* $\mathbf{y} \in E$ *such that* $\mathbf{x} \in X(E, \mathbf{y})$, *then* $\mathbf{x} \in CEF$.

The problems studied in the present paper ar the following:

I. How can we decide whether a given point of E is efficient?

II. How can we obtain the points of $E \setminus X(E, \mathbf{x})$ for any $\mathbf{x} \in E$?

III. Given a function $F : \mathbf{R}^n \to \mathbf{R}$, called supercriterion, list the elements of the set EF in the form $x^1, \ldots, x^r$ such that $F(x^k) \geqslant F(x^{k+1})$, $k = 1, \ldots, r - 1$.

2. Efficiency criteria

THEOREM 1. *If the set* E *is nonempty, then the set* EF *is also nonempty.*

Proof. Define $s : \mathbf{R}^n \to \mathbf{R}$, $s(\mathbf{x}) = \sum_{h \in H} f_h(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^n$. We consider the following problem: maximize the function s on the set E. This problem has an optimal solution $\mathbf{x}^0$. From theorem 1 of GEOFFRION A. M. [4] it follows that $\mathbf{x}^0$ is a an efficient solution for the problem (MP). Therefore $EF \neq \emptyset$.

Remark that a point $\mathbf{x}^0 \in E$ is an efficient solution for (MP) if and only if the system

$$(1) \quad \begin{cases} \mathbf{C}\mathbf{x} \succ \mathbf{C}\mathbf{x}^0 \\ A\mathbf{x} \leqslant \mathbf{b} \\ \mathbf{x} \in B^n \end{cases}$$

is inconsistent, where

$$\mathbf{C} = \begin{pmatrix} c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots \\ c_{s1} & \ldots & c_{sn} \end{pmatrix}.$$

Since the inconsistency of system (1) is difficult to be verified, we give an other efficiency criterium. Let G the set of the solutions of the system

$$(2) \quad \begin{cases} \mathbf{C}\mathbf{x} \geqslant \mathbf{C}\mathbf{x}^0 \\ \mathbf{A}\mathbf{x} \leqslant \mathbf{b} \\ \mathbf{x} \in B^n \end{cases}$$

and define the function $g : \mathbf{R}^n \to \mathbf{R}$,

$$g(\mathbf{x}) = \sum_{j \in J} \Big(\sum_{h \in H} c_{hj}\Big)(x_j - x_j^0) \quad \text{for all } \mathbf{x} \in \mathbf{R}^n.$$

LEMMA 1. *A point* $\mathbf{x}^0 \in E$ *is an efficient solution for* (MP) *if and only if the maximum of the function* g *in* G *is zero.*

Proof. Let $\mathbf{x}^0$ be an efficient solution to (MP). From definition 2 it follows that

$$\sum_{j \in J} c_{hj}\, x_j = \sum_{h \in J} c_{hj}\, x_j^0, \quad h \in H$$

whenever $x \in E$ and $\mathbf{C}\mathbf{x} \geqslant \mathbf{C}\mathbf{x}^0$. Therfore

$$g(\mathbf{x}) = \sum_{j \in J} \Big(\sum_{h \in H} c_{hj}\Big)(x_j - x_j^0) = \sum_{j \in J} \sum_{h \in H} c_{hj}(x_j - x_j^0) = \sum_{h \in H} \sum_{j \in J} c_{hj}(x_j - x_j^0) = 0$$

for all solutions of the system (2). Hence the maximum of the function g on G is zero.

Now, assume that the maximum of the function g on G is equal to 0. Then, for all $\mathbf{x} \in E$ which satisfy

(3) $$\sum_{j\in J} c_{hj}x_j \geqslant \sum_{j\in J} c_{hj}x_j^0, \quad h \in H$$

we have

(4) $$g(x) = \sum_{h\in H}\sum_{j\in J} c_{hj}x_j - \sum_{h\in H}\sum_{j\in J} c_{hj}x_j = 0.$$

Therefore there is no $\mathbf{x} \in E$ such that $\mathbf{Cx} \succ \mathbf{Cx}^0$. Hence $\mathbf{x}^0$ is an efficient solution for (MP).

This lemma is useful to test in the algoritm which will be given in the last section, whether a point $\mathbf{x} \in \mathbf{E}$ is an efficient solution.

Suppose now that the elements of C are rational numbers, i.e.,

$$c_{hj} = \frac{u_{hj}}{v_{hj}},$$

where u_{hj}, v_{hj} are integers and $v_{hj} > 0$ for all $h \in H, j \in J$. For each $h \in H$, let d_h be the least common multiple of the numbers v_{hj}, $j \in J$, and put

(5) $$\tilde{c}_{hj} = c_{hj}d_h, \quad h \in \mathrm{H}, j \in J,$$

$$\tilde{c}_j = \sum_{h\in H} \tilde{c}_{hj}, \quad j \in J,$$

$$\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_s)$$

$$\tilde{C} = \begin{pmatrix} \tilde{c}_{11} \ldots \tilde{c}_{1n} \\ \ldots\ldots\ldots \\ \tilde{c}_{s1} \ldots \tilde{c}_{sn} \end{pmatrix}$$

LEMMA 2. *If the elements of C are rational numbers, then the system*

(6) $$\begin{cases} \mathbf{Cx}^0 \prec \mathbf{Cx} \\ \mathbf{x} \in Z^n \end{cases}$$

is equivalent to the system

(7) $$\begin{cases} \tilde{\mathbf{C}}\mathbf{x} \geqslant \tilde{\mathbf{C}}\mathbf{x}^0 \\ \tilde{\mathbf{c}}\mathbf{x} \geqslant \tilde{\mathbf{c}}\mathbf{x}^0 + 1 \\ \mathbf{x} \in Z^n. \end{cases}$$

Proof. Let $\mathbf{x}$ be a solution of the system (6). Then we have

$$\sum_{j\in J} c_{hj}x_j \geqslant \sum_{h\in J} c_{hj}x_j^0 \quad \text{for all } h \in H,$$

and

$$\sum_{j\in J} c_{hj}x_j > \sum_{j\in J} c_{hj}x_j^0 \quad \text{for at least one } h.$$

By multiplying by $d_h > 0$ we get in view of (5)

(8) $$\sum_{j\in J} \tilde{c}_{hj}x_j \geqslant \sum_{j\in J} \tilde{c}_{hj}x_j^0 \quad \text{for all } h \in H,$$

(9) $$\sum_{j\in J} \tilde{c}_{hj}x_j > \sum_{j\in J} \tilde{c}_{hj}x_j^0 \quad \text{for at least one } h.$$

Adding these s inequalities we get

$$\sum_{j\in J} \tilde{c}_j x_j > \sum_{j\in J} \tilde{c}_j x_j^0$$

Because $\tilde{c}_j$, x_j, $j \in J$ are integers, we have

(10) $$\sum_{j\in J} \tilde{c}_j x_j \geqslant \sum_{j\in J} \tilde{c}_j x_j^0 + 1.$$

From (8) and (10) it follows that $\mathbf{x}$ is a solution of system (7).

Similarly one shows that any solution of system (7) is also a solution of the system (6).

From lemma 2 and from definition 2 we obtain

LEMMA 3. *If the elements of C are rational numbers, then a point* $\mathbf{x}^0 \in E$ *is an efficient solution for* (MP) *if and only if the system*

$$\begin{cases} \tilde{\mathbf{C}}\mathbf{x} \geqslant \tilde{\mathbf{C}}\mathbf{x}^0 \\ \tilde{\mathbf{c}}\mathbf{x} \geqslant \tilde{\mathbf{c}}\mathbf{x}^0 + 1 \\ A\mathbf{x} \leqslant \mathbf{b} \\ \mathbf{x} \in B^n \end{cases}$$

is inconsistent.

The proof is immediately.

3. Dominated points

If $\mathbf{x}^0$ is in E, then the set of the solutions of the system

(12) $$\begin{cases} \mathbf{Cx} \prec \mathbf{Cx}^0 \\ \mathbf{x} \in B^n \\ \mathbf{Ax} \leqslant \mathbf{b} \end{cases}$$

is equal to $X(E, \mathbf{x}^0)$. This remark follows directly from definition 1.

Because it is difficult to find the set of the solutions of the system (12) we give an other criterium. Let $\mathbf{x}^0 \in E$.

LEMMA 4. *The set of the solutions of the system*

$$(13)\quad \begin{cases} \mathbf{C}\mathbf{x} \leqslant \mathbf{C}\mathbf{x}^0 \\ \mathbf{A}\mathbf{x} \leqslant \mathbf{b} \\ \sum_{j\in J}\left(\sum_{h\in H} c_{hj}\right)x_j < \sum_{j\in J}\left(\sum_{h\in H} c_{hj}\right)x_j^0 \\ x \in B^n \end{cases}$$

is equal to $X(E, \mathbf{x}^0)$.

Proof. Let x be a solution of system (13). Then we have

$$(14)\quad \sum_{j\in J} c_{hj}x_j \leqslant \sum_{j\in J} c_{hj}x_j^0 \quad \text{for all } h \in H$$

and

$$(15)\quad \sum_{j\in J}\left(\sum_{h\in H} c_{hj}\right)x_j < \sum_{j\in J}\left(\sum_{h\in H} c_{hj}\right)x_j^0.$$

From (14) and (15) we deduce that there exists $h_0 \in H$ such that

$$(16)\quad \sum_{j\in J} c_{h_0j}x_j < \sum_{j\in J} c_{h_0j}x_j^0.$$

From (14) and (16) it follows that $\mathbf{C}\mathbf{x} \prec \mathbf{C}\mathbf{x}^0$. Therefore $\mathbf{x} \in X(E, \mathbf{x}^0)$.

Let now $\mathbf{x} \in X(E, \mathbf{x}^0)$. Then we have

$$(17)\quad \mathbf{A}\mathbf{x} \leqslant \mathbf{b},$$

$$(18)\quad \sum_{j\in J} c_{hj}x_j \leqslant \sum_{j\in J} c_{hj}x_j^0 \quad \text{for all } h \in H,$$

$$\sum_{j\in J} c_{hj}x_j < \sum_{j\in J} c_{hj}x_j^0 \quad \text{for at least one } h \in H.$$

Adding these s inequalities we get

$$(19)\quad \sum_{j\in J}\left(\sum_{h\in H} c_{hj}\right)x_j < \sum_{j\in J}\left(\sum_{h\in H} c_{hj}\right)x_j^0.$$

From (17), (18) and (19) it follows that $\mathbf{x}$ is a solution of the system (13)

This lemma is used in the algorithm in order to determine the set of points wich are upper dominated by a point $\mathbf{x} \in X$.

From lemma 4 and from lemma 2 we obtain

LEMMA 5. *If the elements of C are rational numbers, then the set of the solutions of the system*

$$(20)\quad \begin{cases} \tilde{\mathbf{C}}\mathbf{x} \leqslant \tilde{\mathbf{C}}\mathbf{x}^0 \\ \mathbf{A}\mathbf{x} \leqslant \mathbf{b} \\ \tilde{\mathbf{c}}\mathbf{x} \leqslant \tilde{\mathbf{c}}\mathbf{x}^0 - 1 \\ \mathbf{x} \in B \end{cases}$$

is equal to $X(E, \mathbf{x}^0)$.

The proof is immediately.

If $x^0 \in B^n$, we put $Q(x^0) = \prod_{j\in J}(1 - 2x_j^0)$ and observe that $Q(x^0) \neq 0$. For future reference we denote

$$H(x^0, x) = \operatorname{sg} Q(x^0)\left(\sum_{j\in J} Q_j(x^0)(x_j - x_j^0) - Q(x^0)\right),$$

where $Q_j(x^0) = \prod_{\substack{k\in J\\ k\neq j}}(1 - 2x_k^0)$.

LEMMA 6. *If* $x^0 \in E$, *then the following equality holds*

$$E\setminus\{\mathbf{x}^0\} = \{\mathbf{x} \in E \mid H(\mathbf{x}^0, \mathbf{x}) \geqslant 0\}.$$

Proof. In order to prove $\{\mathbf{x} \in E \mid H(\mathbf{x}^0, \mathbf{x}) \geqslant 0\} \subseteq E\setminus\{\mathbf{x}^0\}$, it is sufficient to show that $H(\mathbf{x}^0, \mathbf{x}^0) < 0$. Notice that

$$H(\mathbf{x}^0, \mathbf{x}^0) = \operatorname{sg} Q(\mathbf{x}^0)\left(\sum_{j\in J} Q_j(\mathbf{x}^0)(x_j^0 - x_j^0) - Q(\mathbf{x}^0)\right) =$$

$$= -\operatorname{sg} Q(\mathbf{x}^0)\, Q(\mathbf{x}^0) = -|Q(\mathbf{x}^0)| < 0;$$

therefore $\mathbf{x}^0 \notin \{\mathbf{x} \in E \mid H(\mathbf{x}^0, \mathbf{x}) \geqslant 0\}$.

Let $\mathbf{y} \in E\setminus\{\mathbf{x}^0\}$. Without loss of generality we can assume that $\mathbf{y} = (1 - x_1^0, \ldots, 1 - x_k^0, x_{k+1}^0, \ldots, x_n^0)$, where $1 < k \leqslant n$. We have

$$H(\mathbf{y}, \mathbf{x}^0) = \operatorname{sg} Q(\mathbf{x}^0)\left(\sum_{j=1}^{k} Q_j(\mathbf{x}^0)(1 - 2x_j^0) - Q(\mathbf{x}^0)\right) = (k-1)\,|Q(\mathbf{x}^0)| > 0;$$

hence $\mathbf{y} \in \{\mathbf{x} \in E \mid H(\mathbf{x}, \mathbf{x}^0) \geqslant 0\}$. Therefore $E\setminus\{x^0\}$ is a subset of the set $\{\mathbf{x} \in E \mid H(\mathbf{x}, \mathbf{x}^0) \geqslant 0\}$.

Let $\mathbf{x}^0 \in E$. The set $X(E, \mathbf{x}^0)$ has a finit number of points because E is a bounded set. Therefore we can write $X(E, \mathbf{x}^0) = \{\mathbf{x}^1, \ldots, x^r\}$ with $1 \leqslant r \leqslant 2^n$, if $X(E, \mathbf{x}^0) \neq \emptyset$.

Denote by $E(\mathbf{x}^0)$ the set of the solutions of the system

$$(21)\qquad \begin{cases} \mathbf{Ax} \leqslant \mathbf{b} \\ H(\mathbf{x}, \mathbf{x}^k) \geqslant 0,\ k \in \{0, 1, \ldots, r\} \\ \mathbf{x} \in B^n. \end{cases}$$

Then we denote by (MR) the multiple objective program which possesses the constraint set $E(\mathbf{x}^0)$ and the objective function f_h, $h \in H$.

LEMMA 7. *Let* $\mathbf{x}^0$ *be in* E. *A point* $\mathbf{x} \in E$, $\mathbf{x} \neq \mathbf{x}^0$, *is an efficient solution of* (MP) *if and only if it is an efficient solution of problem* (MR).

Proof. Let $\mathbf{x}$ be an efficient solution of (MP). Then we have $\mathbf{Ax} \leqslant \mathbf{b}$, because $\mathbf{x} \in E$. If $r = 0$, then $x \in E \setminus \{\mathbf{x}^0\}$; therefore $H(\mathbf{x}^0, \mathbf{x}) \geqslant 0$. It follows that $\mathbf{x}$ is a solution of the system (21). If $r \geqslant 1$, then $\mathbf{x} \in E \setminus \{\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^r\}$ because $X(E, \mathbf{x}^0) = \{x^1, \ldots, x^r\}$ and $\mathbf{x} \neq \mathbf{x}^0$. Therefore, by lemma 6, we have $H(\mathbf{x}, \mathbf{x}^k) > 0$, $k \in \{0, 1, \ldots, r\}$. It follows that $\mathbf{x}$ is a solution for the system (21). Since $\mathbf{x}$ is an efficient solution for (MP) and $\mathbf{x} \in E(\mathbf{x}^0) \subseteq E$, it follows that $\mathbf{x}$ is an efficient solution for (MR).

Now let x be an efficient solution for (MR). Suppose that it is nonefficient for (MP). Then there is an element $\mathbf{y} \in E \setminus E(\mathbf{x}^0)$ such that $\mathbf{Cy} \succ \mathbf{Cx}$. In view of lemma 6, we have $E(\mathbf{x}^0) = E \setminus \{\mathbf{x}^0, \ldots, x^r\}$; hence $\mathbf{y} \in \{\mathbf{x}^0, \ldots, \mathbf{x}^r\}$. But, if $r \neq 0$ then $\{\mathbf{x}^1, \ldots, x^r\} \subseteq CEF$. Therefore $\mathbf{y} = \mathbf{x}^0$. Since $\mathbf{Cy} = \mathbf{Cx}^0 \succ \mathbf{Cx}$, we have $\mathbf{x} \in X(E, \mathbf{x}^0) = E \setminus (E(\mathbf{x}^0) \cup \{\mathbf{x}^0\})$, a contradiction. It follows that $\mathbf{x}$ is an efficient solution in E.

4. An algorithm for listing the elements of the set EF

Let F be the supercriterion function and let $\mathbf{x}^1$ be an maximum point of F in EF. We call it the first best efficient solution with respect to F.

Definition 3. $\mathbf{x}^2 \in EF$ *is said to be the second best efficient solution with respect to* F, *if* $F(\mathbf{x}^2) = \max\{F(x) \mid x \in EF \setminus \{\mathbf{x}^1\}\}$.

If $\mathbf{x}^2$ is the second best efficient solution with respect to F, then $F(\mathbf{x}^1) \geqslant F(\mathbf{x}^2)$; equality is not excluded. If $F(\mathbf{x}^1) = F(\mathbf{x}^2)$, then $\mathbf{x}^2$ is called an improper second best optimal solution with respect to F. If $F(x^1) > F(\mathbf{x}^2)$, then $\mathbf{x}^2$ is called a proper second best optimal solution with respect to F in EF.

LEMMA 8. *If* x^1 *is the first best efficient solution with respect to* F *in* EF, *then each improper second best efficient solution* $\mathbf{x}^2$ *to* F *on* EF *is a solution of the system*

$$(22)\qquad \begin{cases} F(\mathbf{x}) = F(\mathbf{x}^1) \\ \mathbf{Ax} \leqslant \mathbf{b} \\ H(\mathbf{y}^k, \mathbf{x}) \geqslant 0,\ k \in \{0, \ldots, r\} \end{cases}$$

where $\mathbf{y}^0 = \mathbf{x}^1$ *and* $\mathbf{y}^1, \ldots, \mathbf{y}^r$ *are the elements of* $X(E, \mathbf{x}^1)$, *if* $X(E, \mathbf{x}^1) \neq \emptyset$.

Proof. Since $\mathbf{x}^2$ is an improper second best efficient solution to F on EF, we have $F(\mathbf{x}^2) = F(\mathbf{x}^1)$, $\mathbf{Ax}^2 \leqslant \mathbf{b}$. Suppose that there exists $k \in \{0, \ldots, r\}$ such that $H(\mathbf{y}^k, \mathbf{x}^2) < 0$. Then by lemma 6, we have $\mathbf{x}^2 = y^k$. But $\mathbf{x}^2 \neq \mathbf{x}^1 = \mathbf{y}^0$; therefore $k \in \{1, \ldots, r\}$. Because $\{y^1, \ldots, y^r\} = X(E, \mathbf{x}^1)$, it follows that $\mathbf{x}^2 \in X(E, \mathbf{x}^1) \subseteq CEF$, which contradicts the efficiency of $\mathbf{x}^2$. Therefore $H(\mathbf{y}^k, \mathbf{x}^2) \geqslant 0$ for all $k \in \{0, \ldots, r\}$. It follows that $\mathbf{x}^2$ is a solution of the system (22).

DESCRIPTION OF THE ALGORITHM

Now we are going to describe an algorithm, based on lemmas 1, 4, 6, 7, which allows us to enumerate succesively the second best efficient solutions of F in EF and to order in this way the set EF with respect to F.

Step 0. Put $E_0 = E$, $E_1 = E$, $i = 1$, $k = 0$, $EF = \emptyset$.

Step 1. If $E_i = \emptyset$, then the algorithm stops. If $E_i \neq \emptyset$, find a maximum point $\mathbf{x}^i$ of the function F in E_i. In order to decide whether E_i is empty or not and to find $\mathbf{x}^i$, the methods given in [5], [6] and [8] can be used.

Step 2. Using lemma 1, test whether $\mathbf{x}^i$ is an efficient solution. If it is go to step 3. Otherwise, put $E_i := \{x \in E_i \mid H(\mathbf{x}^i, \mathbf{x}) \geqslant 0\}$ and go to step 1.

Step 3. Put $k := k + 1$, $\mathbf{z}^k = \mathbf{x}^i$ and $EF = EF \cup \{\mathbf{z}^k\}$. Using lemma 4, find the set $X(E, \mathbf{x}^i)$ and put $E_{i+1} := \{\mathbf{x} \in E_i \mid H(y, \mathbf{x}) \geqslant 0$ for all $\mathbf{y} \in X(E_i, \mathbf{x}^i) \cup \{\mathbf{x}^i\}\}$.

If the system

$$(23)\qquad \begin{cases} F(\mathbf{x}) = F(\mathbf{x}^i) \\ \mathbf{x} \in E_{i+1} \end{cases}$$

is inconsistent, go to step 7. If the system (23) is consistent, then denote by $X(E_i, \mathbf{x}^i) = \{\mathbf{y}^1, \ldots, \mathbf{y}^r\}$ the set of its solutions.

Step 4. Put $j = 1$.

Step 5. Using lemma 1 test whether y^j is an efficient solution. If it is, put $E_{i+1} := \{x \in E_{i+1} \mid H(y^j, x) \geqslant 0\}$, $k = k + 1$, $z^k = y^j$ and $EF = EF \cup \{z^k\}$.

Using lemma 4, find the set $X(E_{i+1}, y^j)$; put

$$E_{i+1} := \{x \in E_{i+1} \mid H(y, x) \geqslant 0 \text{ for all } \mathbf{y} \in X(E_{i+1}, \mathbf{y}^j)\}$$

and go to step 6. If $\mathbf{y}^j$ is not an efficient solution, go to step 6.

Step 6. Increase j with 1. If $j \leqslant r$, go to step 5. If $j > r$, go to step 7.

Step 7. Increase i with 1 and return to step 1.

THEOREM 2. (a) *There exists a natural number* i, $i \leqslant 2^n + 1$, *such that* $E_i = \emptyset$.

(b) *The set EF given by the algorithm contains all efficient solutions of* (MP).

(c) *If* $EF \neq \emptyset$ *and* $k > 1$, *we have* $F(\mathbf{z}^j) \geqslant F(\mathbf{z}^{j+1})$, $j = 1, \ldots, k-1$.

Proof. (a) If $E_1 = \emptyset$, then $i = 1$ and the affirmation (a) is proved. If $E_1 = \emptyset$ then from step 2, 3 and 5, we have $E_{j+1} \subset E_j$ for all $j \in \{1, \ldots, i-1\}$ and $E_1 \subset E$. Then card $(E_{j+1}) \leqslant$ card $(E_j) - 1$ for all $j \in \{1, \ldots, i-1\}$. Because card $E \leqslant 2^n$, it follows that, in the worst case, we have $E_j = \emptyset$ for $j = 2^n + 1$.

(b) Let $\mathbf{x}^0 \in E \setminus EF$. Then there is an index $j \in \{0, \ldots, i-1\}$ such that $\mathbf{x}_j \in E_j$ and $\mathbf{x}^0 \notin E_{j+1}$. Four cases may appear:

1) If $\mathbf{x}^0$ is a maximum point in E_j, determined in step 1, then step 2 and lemma 1 imply that $\mathbf{x}^0$ is not an efficient solution.

2) If $\mathbf{x}^0 \in X(E_j, \mathbf{x}^j)$, then from remark 1 we have $\mathbf{x}^0 \in CEF$.

3) If $\mathbf{x}^0 \in X(E_j, x^j)$, then from step 5 it follows that $\mathbf{x}^0$ is not an efficient solution.

4) If there exists $k \in \{1, \ldots, r\}$ such that $\mathbf{x}^0 \in X(E_{j+1}, y^k)$, then from remark 1 of section 1, we have $x^0 \in CEF$.

Therefore, in all cases, x^0 is not an efficient solution.

Let $x \in EF$. From steps 2, 5 and from lemma 7 it follows that x is an efficient solution.

(c) Let $EF \neq \emptyset$ and $k > 1$. Let $h \in \{1, \ldots, k-1\}$. We have two cases:

— There exists j such that $\mathbf{z}^h \in E_j$, $\mathbf{z}^h \notin E_{j+1}$ and $\mathbf{z}^{h+1} \notin E_{j+1}$, $\mathbf{z}^{h+1} \in E_j$; then by step 3 we have $F(\mathbf{x}^h) = F(\mathbf{x}^{h+1})$;

— There exists j such that $\mathbf{z}^h \in E_j$, $\mathbf{z}^h \notin E_{j+1}$ and $\mathbf{z}^{h+1} \in E^{j+1}$; then it follows that $F(\mathbf{x}^h) \geqslant F(\mathbf{x}^{h+1})$, because $E_{j+1} \subset E_j$ and $\mathbf{z}^h$ is a maximum point of F on E_j and $\mathbf{z}^{h+1}$ is a maximum point of F on E_{j+1};

Remark 2. *If the elements of C are rational numbers, then instead of lemma* 1, *in step* 2, *and* 5, *respectively lemma* 4 *in step* 3, *we may use lemma* 3 *respectively lemma* 5.

LEMMA 10 [see 10]. *Let* $p = (p_1, \ldots, p_s)^T \in \mathbf{R}^s$ *and* $p_h > 0$, $h \in H$. *If*

$$F(\mathbf{x}) = \mathbf{p}^T C \mathbf{x} \text{ for all } \mathbf{x} \in \mathbf{R}^n$$

and $\mathbf{x}^0$ *is a maximum point of F in E, then* $\mathbf{x}^0 \in EF$.

From lemma 10 we have the following remark.

Remark 3. *If the function F is a positive combination of the objective functions* f_h, $h \in H$, i.e. $F(\mathbf{x}) = \sum_{h \in H} p_h f_h(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbf{R}^n$, *where* $p_h > 0$, *then we can drop step* 2 *of the algorithm going directly from step* 1 *to step* 3.

Also, we can drop the test in step 5, *because from lemma* 10 *we always conclude that* $\mathbf{y}^j \in X(E_i, \mathbf{x}^i)$ *is an efficient solution.*

This algorithm is a theoretical algortihm. From this we can obtain the good practical algorithms.

5. Numerical example

To illustrate the algorithm we consider the following objective functions $f_i : B^3 \to \mathbf{R}$, $i = 1, 2, 3$ and

$$f_1(x) = 4x_1 + x_2 + 2x_3 \text{ for all } x = (x_1, x_2, x_3)^T \in B^3$$

$$f_2(x) = x_1 + 3x_2 - x_3 \text{ for all } x = (x_1, x_2, x_3)^T \in B^3$$

$$f_3(x) = -x_1 + x_2 + 4x_3 \text{ for all } x = (x_1, x_2, x_3)^T \in B^3$$

and let

$$E = \{(x_1, x_2, x_3)^T \in B^3 \mid x_1 + x_2 + x_3 \leqslant 3,\ 2x_1 + 2x_2 + x_3 \leqslant 4,\ x_1 - x_2 \leqslant 0\}.$$

Choose as supercriterion the function $F : B^3 \to R$,

$$F(x) = 4x_1 + 5x_2 + 5x_3 \text{ for all } x = (x_1, x_2, x_3)^T \in B^3.$$

Since F is a positive combination of the functions f_1, f_2, f_3 and the elements of C are natural numbers, we can use the remarks 2 and 3.

Step 0. Put $E_0 = E$, $E_1 = E$, $EF = \emptyset$, $i = 1$, $k = 0$.

Step 1. The maximum point of F on E_1 is found to be $\mathbf{x}^1 = (0, 1, 1)^T$. In view of remark 3, it is an efficient solution.

Step 3. Put $k = 1$, $z^1 = (0, 1, 1)$ and $EF = \emptyset \cup \{(0, 1, 1)^T\} = \{(0, 1, 1)^T\}$. For $X(E_1, \mathbf{x}^1)$ we get $X(E, \mathbf{x}^1) = \{(0, 0, 0)^T, (0, 0, 1)^T\}$. Observe that

$$E_2 = \{\mathbf{x} \in E_1 \mid x_1 - x_2 - x_3 \geqslant -1,\ x_1 + x_2 + x_3 \geqslant 1,\ -x_1 - x_2 + x_3 \leqslant 0\}.$$

Since the system

$$\begin{cases} F(\mathbf{x}) = 10 \\ \mathbf{x} \in E_2 \end{cases}$$

is inconsistent, we go to step 7.

Step 7. Increase i with 1 and we return to step 1.

Step 1. The maximum point of F on E_2 is determined as $\mathbf{x}^2 = (1, 1, 0)^T$. In view of remark 3, it is an efficient solution.

Step 3. Put $k = 2$, $z^2 = (1, 1, 0)^T$ and $EF = \{(0, 1, 1)^T, (1, 1, 0)^T\}$. Since $X(E_2, x^2) = \emptyset$, we put $E_3 = \{x \in E_2 \mid -x_1 - x_2 - x_3 \geq -1\}$. The system

$$\begin{cases} F(x) = 9 \\ x \in E_3 \end{cases}$$

is inconsistent. We go to step 7.

Step 7. Increase i with 1 and we return to step 1.

Step 1. The maximum point of F on E_3 is found to be $x^3 = (0, 1, 0)^T$. Because it is an efficient solution, go to step 3.

Step 3. Put $k = 3$, $z^3 = (0, 1, 0)^T$ and $EF = \{(0, 1, 1)^T, (1, 1, 0)^T\} \cup \cup \{(0, 1, 0)^T\}$. Since $X(E_3, x^3) = \emptyset$, we put $E_4 = \{x \in E_3 \mid x_1 - x_2 + x_3 \geq 0\}$ and because the system

$$\begin{cases} F(x) = 5 \\ x \in E_4 \end{cases}$$

is inconsistent, go to step 7.

Step 7. Increase i with 1 and go to step 1.

Step 1. Since $E_4 = \emptyset$ the algorithm stops.

The set of efficient points is $EF = \{z^1 = (0, 1, 1)^T, z^2 = (1, 1, 0)^T, z^3 = (0, 1, 0)^T\}$.

There are certainly several other aspects and questions to be explored in linear multiple objective problems with zero-one variables. This paper is an attempt to establish underlying results and to give a procedure for listing the elements of the efficient set in the form $x^1, \ldots, x^r$ such that if F is a supercriterion function, then $F(x^k) \geq F(x^{k+1})$ for each $k \in \{1, \ldots, r-1\}$.

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