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ON SOME SEQUENCE TO FUNCTION TRANSFORMATIONS

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1. Suppose that $q_n \geqslant 0$ and $q_n \neq 0$ for infinitely many valeus of n. We shall let x and z stand, throughout, for a real and complex number respectively. Let r denote the radius of convergence of the power series $\sum_{n=0}^{\infty} q_n z^n$ $(r \leqslant \infty)$. The analytic function represented by this power series for $|z| < r$ is given by

$$(1.1) \qquad q(z) = \sum_{n=0}^{\infty} q_n z^n \ (|z| < r).$$

Given an infinite series Σa_n with partial sum $\{s_n\}$ we say that the method (J, q) is applicable to $\sum_{n=0}^{\infty} a_n$, if the series $\sum_{n=0}^{\infty} q_n s_n z^n$ converges for $|z| < r$, say to $q_s(z)$, and the sequence to function transformation $J^q(x) = q_s(x)/q(x)$ exists for $0 < x < r$. Further, if $J^q(x) \to l$ $(x \to r-)$, then the series Σa_n is said to be summable (J, q) to l. It is said to be absolutely summable (J, q) or summable $|J, q|$ if $J^q(x) \in BV(0, r)$ i.e. $\int_0^r |d J^q(x)| < \infty$. BORWEIN [1] has shown that the method (J, q) is regular if and only if $q(x) \to \infty$ as $x \to r-$. BORWEIN [1, 2, 3] considered the inclusion relations between (J, p) and (J, q) methods of summability. Das [7] has obtained inclusion relation between $|J, p|$ and $|J, q|$ methods.

As well-known particular cases of the (J, q) method, we have the Abel method (A) when $q_n = (n+1)^{-1}$ (see [9, 5], the method A_α, when $q_n = \binom{n+\alpha}{\alpha}$ (see [1, 5] (A_0 is the same as the Abel method) and the method B_α when $q_n = (\Gamma(n+\alpha+1))^{-1}$ (B_0 is the same as the Borel method) (see [9], p. 222).

A real method of summation T is totally regular if $s_n \to s$ implies that T-limit of $s_n \to s$ for all finite and infinite s as $n \to \infty$. It is known that a necessary and sufficient condition for a real triangular matrix transformation to be totally regular is that it should be regular and positive (see [9], and [10] for a general result on the subject).

Throughout the paper we shall use the following notations:

For two summability processes A and B, $A \subset B$ will mean that all sequences (series) summable (A) are summable (B).

c will denote the space of convergent sequences.

$\Sigma_n a_n \in (A)$ will mean that the series $\Sigma_n a_n$ is summable by the method (A). If in this statement we replace (A) by c then it will mean that the series $\Sigma_n a_n$ is a convergent series.

$\varepsilon_n \in (A, B)$ will stand for the statement that „summability (A) of Σa_n implies summability (B) of $\Sigma a_n \varepsilon_n$".

2. Eventhough for a regular (J, q) method the summability field of (J, q) includes those of $|J, q|$ and convergence, it is not clear if either of these include the other for general $\{q_n\}$. However, when $q_n = 1$, $n = 1, 2, \ldots$, so that (J, q) is the Abel metod A, WHITTAKER [16], by an example suggested by J. E. LITTLEWOOD showed that a Fourier series may converge at a point without being summable $|A|$, while PRASAD [14] constructed an example of a Fourier series which is summable $|A|$ at a point without being convergent at that point. Thus we can conclude that the properties of convergence and summability $|A_\alpha|$ of infinite series are independent of each other atleast for $\alpha = 0$. This however raises the following problem: Does there exist any $\alpha > -1$ for which the properties of convergence and $|A_\alpha|$ are not independent of each other. We do not know the answer.

When $q_n = (n!)^{-1}$, so that (J, q) is the Borel method (B), we show that the properties of convergence and summability $|B|$ for an infinite series are independent of each other. Our remak is supported by

PROPOSITION 1. (i) *There is a series summable* $|B|$ *which is not convergent.*

(ii) *There is a series which is convergent but not summable* $|B|$.

Proof. (i). Consider the series for which the nth partial sum is $(-1)^n$. This series is not convergent but its Borel transform $B(x) = e^{-x}\sum_{n=0}^{\infty}(-1)^n x^n/n! = e^{-2x} \in BV\,[0, \infty)$. Thus the series is summable $|B|$.

(ii) Let $a_n = 0$ $(n = 0)$, and $a_n = (\sin nt)/n$ $(n = 1, 2, \ldots)$. The series Σa_n converges for all t. After simplification it can be seen that if this series were summable $|B|$ for $t = y$ then the integral

$$I \equiv \int_{n=0}^{\infty}\left| e^{-x}\sum_{n=0}^{\infty}(x^n \sin(n+1)y)/(n+1)!\right| dx \tag{2.1}$$

will be convergent. Denoting the term inside the modulus sign by $K(x, y)$, we have

$$k(x, y) = \operatorname{Im}\left| e^{-x}\sum_{n=0}^{\infty}(x^n e^{i(n+1)y}/(n+1)!) = \right.$$

$$= x^{-1}e^{-2x\sin^2 y/2}\sin(x\sin y) - x^{-1}e^{-x}.$$

Thus

$$I \geqslant \int_0^{\infty} x^{-1}e^{-x}dx - \int_0^{\infty}|x^{-1}e^{-2x\sin^2 y/2}\sin(x\sin y)|dx. \tag{2.2}$$

Choose $\delta > 0$ so small that $\sin(x \sin y)$ is non-negative. Then the second integral of (2.2) is not greater than

$$\sin y\int_0^{\delta}\{e^{-2x\sin^2 y/2}\sin(x\sin y)/x\sin y\}\,dx + \int_{\delta}^{\infty}x^{-1}e^{-2x\sin^2 y/2}dx =$$

$$= O\left(\int_0^{\delta}e^{-2x\sin^2 y/2}dx\right) + O\left(\delta^{-1}\int_{\delta}^{\infty}e^{-2x\sin^2 y/2}dx\right) = O((\sin^2 y/2)^{-1}).$$

Choosing y to be different from an even multiple of π, we have the above integral bounded. But the divergence to infinity of the first integral of (2.2) shows that I is divergent. Hence we establish the assertion.

3. In view of § 2 a natual question is to obtain necessary and sufficient conditions on a sequence $\{\varepsilon_n\}$ such that $\Sigma a_n \varepsilon_n$ is either summable $|B|$ or $|A_\alpha|$ $(\alpha > -1)$ whenever Σa_n is a convergent series. Along this line is the following result:

THEOREM A [15], $\varepsilon_n \in (c, |A|)$ *if and only if*

$$\Sigma|\Delta\varepsilon_n| < \infty, \tag{3.1}$$

$$\Sigma|\varepsilon_n|\,n^{-1} < \infty. \tag{3.2}$$

We shall first obtain ε_n such that $\varepsilon_n \in (c, |J, q|)$ and then obtain results for $|A_\alpha|$ and $|B|$ by assigning particular values to q_n. Our results will be the following theorems:

THEOREM 1. *Let $q_n \geqslant 0$ and (J, q) method be totally regular. Then $\varepsilon_n \in$ $\in (c, |J, q|)$ only if*

(3.3) $$\Sigma |\triangle \varepsilon_n| < \infty,$$

and

(3.4) $$\Sigma |\varepsilon_n| \, q_n |\varkappa_n| < \infty,$$

hold, where

(3.5) $$\varkappa_n = \int_0^r \left\{ \frac{d}{dx} (x_n / q(x)) \varphi(x) \right\} dx$$

for every measurable, essentially bounded real function $\varphi(x)$.

THEOREM 2. *Let $q_n \geqslant 0$. Then the sufficient conditions for $\Sigma a_n \varepsilon_n \in$ $\in |J, q|$ whenever $s_n \equiv \sum_{k=0}^{n} a_k = O(1)$ are* (3.3) and

(3.6) $$\Sigma |\varepsilon_n| q_n |\psi_n| < \infty,$$

where

(3.7) $$\psi_n = \int_0^r \frac{d}{dx} \left| (x^n / q(x)) \right| dx.$$

Remark. $\varkappa_n$ of (3.5) always exists for $q_n \geqslant 0$ (see [8]).

4. We shall need the following lemmas:

LEMMA 1 ([6], lemma 8). *If $\Sigma g_n(x) s_n$ converges for $0 < x < r$ and its sum tends to a limit as $x \to r - 0$ whenever s_n is convergent, then there are numbers M, X such that $\Sigma |g_n(x)| \leqslant M$ for $X < x < r$.*

LEMMA 2 ([13], see also [15]). *If a sequence $\{p_n\}$ of elements in a Banach space B has the property that there is a number H such that $\left\| \sum_{n=0}^{k} \pm p_n \right\| \leqslant H$ for each k and every set of signs $\pm$, then $\sum_{n=0}^{\infty} |f(p_n)| < \infty$ for every linear functional f on B.*

LEMMA 3. *Let $q_n \geqslant 0$ and (J, q) method be totally regular. Then $\Sigma a_n \in c$ and $\Sigma a_n \varepsilon_n \in (J, q)$ implies $\varepsilon_n = O(1)$.*

Proof. If under the hypotheses of the lemma ε_n is not bounded, then $\limsup_{n\to\infty} |\varepsilon_n| = +\infty$. Then there exists a sequence of positive non-decreasing sequence of positive integres n_ν, $\nu = 1, 2, \ldots$ such that $|\varepsilon_{n_\nu}| > \nu^2$. Choose a_n such that $a_n = 0$ $(n \neq n_\nu)$ and $a_n = \nu^{-2} \operatorname{sgn}^{-1} \varepsilon_{n_\nu}$ $(n = n_\nu)$. Thus $\Sigma |a_n| \leqslant \sum_\nu \nu^{-2} < \infty$. But when $n = n_\nu$, $a_n \varepsilon_n = a_{n_\nu} \varepsilon_{n_\nu} = \nu^{-2} |\varepsilon_{n_\nu}| > 1$, and so the series $\Sigma a_n \varepsilon_n$ diverges to $+\infty$. Since (J, q) is assumed to be a totally a regular method $\Sigma a_n \varepsilon_n$ is non-summable (J, q). Thus $\varepsilon_n = O(1)$.

LEMMA 4. *Let $q_n \geqslant 0$ and (J, q) method be totally regular. Then $\varepsilon_n \in$ $\in (c, (J, q))$ only if (3.3) holds.*

Proof. Writing $J(x)$ for the (J, q) mean of the series $\Sigma a_n \varepsilon_n$ we have

(4.1) $$J(x) = q_s(x) / q(x),$$

where

(4.2) $$q_s(x) = \sum_{n=0}^{\infty} q_n x^n \sum_{k=0}^{n} a_k \varepsilon_k.$$

Since, by hypothesis and Lemma 3, a_n is convergent and $\varepsilon_n = O(1)$, we have

$$\sum_{n=0}^{\infty} \left| q_n x^n \sum_{k=0}^{n} a_k \varepsilon_k \right| = O\left(\sum_{n=0}^{\infty} n q_n |x|^n \right) = O(1),$$

whenever $|x| < r$ since the radius convergence of $\Sigma q_n x^n$ is r. Thus by change of order of summation in (4.2), we obtain

(4.3) $$q_s(x) = \sum_{k=0}^{\infty} a_k \varepsilon_k \sum_{n=k}^{\infty} q_n x^n.$$

By Abel's transformation

(4.4) $$q_s(x) = \sum_{n=0}^{\infty} s_k \triangle_k \left(\varepsilon_k \sum_{k=0}^{n} q_n x^n \right)$$

provided that

(4.5) $$\lim_{m\to\infty} (s_m \varepsilon_m q_m x^m) / q(x) = 0.$$

But by hypothesis s_n converges, by Lemma 3, since $\varepsilon_n = O(1)$ and since $\Sigma q_n x^n$ is convergent for $0 \leqslant x < r$, $q_n x^n \to 0$ as $n \to \infty$. Hence (4.5) holds and therefore (4.4) is valid. Now write (4.1) in the from

(4.6) $$J_s(x) = \Sigma g_k(x) s_k$$

where

(4.7) $$g_k(x) = (q(x))^{-1} \triangle_k \left(\varepsilon_k \sum_{n=k}^{\infty} q_n x^n \right).$$

Since, by hypothesis, $J(x)$ exists for $0 \leqslant x < r$ and $J(x) \to a$ limit as $x \to r-$ whenever $\{s_n\}$ converges, by Lemma 2, there exists numbers M and X such that

$$(4.8)\qquad \Sigma_k |g_k(x)| \leqslant M \quad (\text{for } X < x < r \text{ and for all } n).$$

It folows that

$$(4.9)\qquad \limsup_{x\to r-} \Sigma |g_k(x)| \leqslant M.$$

But since, as $x \to r-$, $\Sigma q_n x^n \to \infty$, and so $\sum_{n=k}^{\infty} q_n x^n/q(x) = 1 - \left(\sum_{n=0}^{k-1} q_n x^n/q(x)\right)$ which tends to 1, it follows from (4.7) that $g_k(x) \to \triangle_k \varepsilon_k$. Hence

$$(4.10)\qquad |\triangle_k \varepsilon_k| = \Sigma_k |\lim_{x\to r} g_k(x)| \leqslant \liminf_{x\to r} |g_k(x)| \leqslant \limsup_{x\to r} |g_k(x)| \leqslant M,$$

by (4.9).

5. *Proof of theorem* 1. Since $|(J, q)| \subset (J, q)$, necessity of (3.3) follows from Lemma 4. In proving the necessity of (3.4) we shall use notations of Lemma 4 for $J(x)$ etc. without restatement.

Since $J(x)$ exists for $0 \leqslant x < r$ and so is differentiable in $[0, r)$, we obtain by straightforward calculation,

$$(5.1)\qquad J'(x) = -\sum_{n=0}^{\infty} a_n \varepsilon_n \frac{d}{dx}\left(\left(\sum_{k=0}^{n-1} q_k x^k\right)/q(x)\right).$$

By Abel's transformation,

$$(5.2)\qquad J'(x) = -\sum_{n=0}^{\infty} s_n \triangle_n \left(\varepsilon_n \frac{d}{dx}\left\{\left(\sum_{k=0}^{n-1} q_k x^k\right)/q(x)\right\}\right)$$

provided that

$$(5.3)\qquad \lim_{m\to\infty} s_m \varepsilon_{m+1} \frac{d}{dx}\left\{\left(\sum_{k=0}^{m} q_k x^k\right)/q(x)\right\} = 0.$$

But since $\{s_n\} \in l_\infty$, $\{\varepsilon_n\} \in l_\infty$ and $\frac{d}{dx}\left\{\sum_{k=0}^{m} q_k x^k/q(x)\right\} \to$

$$\to \{(q'(x)q(x) - q'(x)q(x))/(q(x))^2\},$$

as $m \to \infty$, for $0 \leqslant x < r$, condition (5.3) is satisfied. Now from (5.2), we have

$$(5.4)\qquad J'(x) = -\sum_{n=0}^{\infty} s_n \varepsilon_n q_n \frac{d}{dx}(x^n/q(x)) - s_n \triangle \varepsilon_n \frac{d}{dx}\left(\left(\sum_{k=0}^{n} q_k x^k\right)/q(x)\right) \equiv$$

$$\equiv J_1(x) + J_2(x).$$

It con be checked that

$$(5.5)\qquad \frac{d}{dx}\left(\left(\sum_{k=0}^{n} q_k x^k\right)/q(x)\right) = \left(\sum_{k=0}^{\infty} V_k x^k\right)/q(x)^2$$

where

$$V_k = q_k q_1 + 2q_2 q_{k-1} + \ldots + n q_n q_{k-n+1} - (k+1) q_{k+1} q_0 - k q_k q_1 - \ldots -$$
$$- (k - n + 1) q_n q_{k-n+1} = -(k+1) q_{k+1} q_0 - (k-1) k q_k q_1 - \ldots -$$
$$- (k - 2n + 1) q_{k-n+1} q_n,$$

it being understood that $q_r = 0$ if r is a negative integer. Now we separately consider the cases $0 \leqslant k \leqslant n$, $n < k \leqslant 2n$, $k > 2n$. In each case it can be checked that $V_k \leqslant 0$ for all n and for $0 \leqslant x < r$ whenever $q_n \geqslant 0$ (see McFadden [11]).

Hence it follows from (5.5) that

$$(5.6)\qquad \int_0^r \left|\frac{d}{dx}\left(\left(\sum_{k=0}^{n} q_k x^k\right)/q(x)\right)\right| dx = -\left(\sum_{k=0}^{n} q_k x^k\right)/q(x) = 1,$$

since $q(x) \to \infty$ as $x \to r-$.

Now from (5.4), we have by (5.6)

$$\int_0^r |J_2(x)|\, dx \leqslant \sum_{n=0}^{\infty} |s_n| |\triangle \varepsilon_n| \int_0^r \left|\frac{d}{dx}\left(\sum_{k=0}^{n} q_k x^k\right)/q(x)\right| dx =$$

$$= \sum_{n=0}^{\infty} |s_n| |\triangle \varepsilon_n| \leqslant K \sum_{n=0}^{\infty} |\triangle \varepsilon_n| \leqslant K,$$

by (3.3) and the fact that $s_n = O(1)$.

We are given $\int_0^r |J'(x)|\, dx < \infty$. Since $\int_0^r |J_2(x)|\, dx < \infty$, it follows from (5.4) that

$$(5.7)\qquad \int_0^r |J_1(x)|\, dx < \infty,$$

which is the same as

$$\int_0^r \left|\sum_{n=0}^{\infty} s_n \varepsilon_n q_n \{(d/dx)(x^n/q(x))\} dx\right| < \infty,$$

for every convergent sequence s_n. But (5.7) holds (see [15], lemma 2) if and only if

$$\int_0^r \left| \Sigma s_n \varepsilon_n q_n \frac{d}{dx} (x^n/q(x)) \right| dx \leqslant H \overline{bd}|s_n| \tag{5.8}$$

for some absolute positive constant H. In particular, (5.8) implies that

$$\int_0^r \left| \sum_{n=0}^{k} \pm \varepsilon_n q_n \frac{d}{dx} (x^n/q(x)) \right| \leqslant H \tag{5.9}$$

for each k and every sequence of signs. Hence, by Lemma 3, we have

$$\sum_{n=0}^{\infty} |\varepsilon_n| q_n \left| \int_0^r \varphi(x) \frac{d}{dx} (x^n/q(x)) \right| dx < \infty \tag{5.10}$$

for every bounded, real function $\varphi(x)$.

This completes the proof of Theorem 1.

Proof of theorem 2. We are given that $s_n = O(1)$. Since

$\sum_{k=0}^{n} a_k \varepsilon_k = \sum_{k=0}^{n-1} \triangle\varepsilon_k s_k + \varepsilon_n s_n$, we have $\sum_{k=0}^{n} a_k \varepsilon_k \leqslant \sup |s_k| \sum_{k=0}^{n-1} |\triangle \varepsilon_k| +$ $+ O(1) = O(1)$, so that $q_s(x) = O(1) \sum_{n=0}^{\infty} q_n x^n$. Hence $q_s(x)$ exists for $|x| < r$. Since $q(x) \neq 0$ for $|x| < r$, it follows that $J(x)$ exists for $|x| < r$ as a power series expansion and therefore $J(x)$ is differentiable in $[0, r)$. Now we have

$J'(x) = J_1(x) + J_2(x)$ as in (5.4). But whenever $\Sigma |\triangle \varepsilon_n| < \infty$, we have, as before $\int_0^r |J_2(x)|\, dx < \infty$. We have only to show that $\int_0^r |J_1(x)|\, dx < \infty$. Now

$$\int_0^r |J_1(x)|dx \leqslant \sum_{n=0}^{\infty} |s_n \varepsilon_n q_n| \int_0^r \left| \frac{d}{dx} (x^n/q(x)) \right| dx \leqslant k \sum_{n=0}^{\infty} |\varepsilon_n| q_n \psi_n \leqslant k.$$

This completes the proof of Theorem 2.

6. In this section we apply Theorem 1 and Theorem 2 to obtain results for summability method $|A_\alpha|$. In this case $r = 1$, $q(x) = (1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A_n^\alpha x^n$.

THEOREM 3. *Σa_n is convergent implies $\Sigma a_n \varepsilon_n \in |A_\alpha|$ $(\alpha > -1)$ if and only if* (3.1) *and* (3.2) *hold.*

Proof. Sufficiency. In this case

$$\frac{d}{dx} (x^n/q(x)) = (1-x)^\alpha x^{n-1}((1-x)n - x(1+\alpha)).$$

Thus

$$\frac{d}{dx} (x^n/q(x) = \begin{cases} \geqslant 0 \ (x \leqslant n/(\alpha+1+n)\,; \\ < 0 \ (x > n/(\alpha+1+n). \end{cases}$$

If we write $f(x)$ for $(x^n/q(x))$, we have,

$$\int_0^1 \left| \frac{d}{dx} (x^n/q(x)) \right| dx = \int_0^{n/(n+\alpha+1)} \frac{d}{dx} (x^n/q(x)) dx - \int_{n/(n+\alpha+1)}^1 \frac{d}{dx} (x^n/q(x)) dx =$$

$$= f(n/(\alpha+1+n)) - f(0) - f(1) + f(n/(\alpha+1+n)) = 2f(n/(\alpha+1+n)) =$$

$$= (n/(\alpha+1+n))^n (\alpha+1)^{\alpha+1} (n+\alpha+1)^{-\alpha-1} = O(n^{-\alpha-1}).$$

Hence from Theorem 2 we obtain

$$\sum_{n=0}^{\infty} |\varepsilon_n| q_n \psi_n = O\left(\sum |\varepsilon_n|/n^{\alpha+1}\right) A_n^\alpha\big) = O\left(\sum n^{-1} |\varepsilon_n|\right) = O(1),$$

by the hypothesis.

Necessity. We first observe that we do not impose any additional restriction by assuming that (3.4) holds for every bounded complex function $\varphi(x)$. We next set $\varphi(x) = (1-x)^i$ $(i = \sqrt{-1})$ in (3.4). Then the integral is given by

$$\left| \int_0^1 (1-x)^i \frac{d}{dx} ((1-x^n)^{\alpha+1} x^n)) dx \right| = \left| i \int_0^1 (1-x)^{\alpha+i} x^n dx \right| =$$

$$= | i\Gamma(1+\alpha+i)\Gamma(n+1)/\Gamma(n+2+\alpha) | \cong (1+i+\alpha) | n^{\alpha+1}.$$

Hence

$$\Sigma_n |\varepsilon_n| A_n^\alpha \left| \int_0^1 (1-x)^i \frac{d}{dx} \{(1-x)^{\alpha+1} x^n\} dx \right| < \infty \quad \text{i.e. } \Sigma_n (|\varepsilon_n|/n) < \infty.$$

Thus the proof of the theorem is complete.

7. In this section we apply Theorem 1 and Theorem 2 to absolute Borel summability. In this case $q(x) = e^x = \sum_{n=0}^{\infty} (x^n/n!)$, and $r = \infty$. Our result is the following:

THEOREM 4. $\varepsilon_n \in (c, |B|)$ *if and only if*

$$\Sigma_n |\triangle \varepsilon_n| < \infty. \tag{7.1}$$

and

$$\Sigma_n \{|\varepsilon_n| / n^{\frac{1}{2}}\} < \infty. \tag{7.2}$$

Proof. Let $f(x) = x^n/q(x) \equiv e^{-x} x^n$.

Hence $f'(x) = -e^{-x} x^n + n x^{n-1} e^{-x} = e^{-x} x^{n-1}(n - x)$. So

$$f'(x) = \begin{cases} \geqslant 0 \ (x \leqslant n); \\ < 0 \ (x > n). \end{cases} \tag{7.3}$$

Sufficiency. In view of (7.3) we find that

$$\int_0^\infty |f'(x)| dx = \left(\int_0^n + \int_n^\infty \right) |f'(x)| dx = \int_0^n f'(x) dx - \int_n^\infty f'(x) dx =$$

$$= f(n) - f(0)) - f(\infty)) - f(n)) = 2f(x) = 2n^n e^{-n},$$

since $f(0) = f(\infty) = 0$. Now

$$\Sigma_n |\varepsilon_n| \, q_n \, \psi_n = 2 \Sigma_n (|\varepsilon_n| \, n^n e^{-n}/n) \cong \Sigma_n (|\varepsilon_n| / n^{\frac{1}{2}}) < \infty.$$

Necessity. $x_n = \int_0^\infty f'(x) \varphi(x) dx$ Choose $\varphi(x) = 1$ $(x \leqslant n)$, and $\varphi(x) = -1$ $(x > n)$. Then

$$x_n = \int_0^\infty f'(x) \varphi(x) dx = \left(\int_0^n - \int_n^\infty \right) f'(x) \varphi(x) dx = 2n^n e^{-n}.$$

Substituting the value in (3.4) we see the necessity part of (7.2).

8. If one sets $q(x) = \log(1/(1-x))$ and $r = 1$ then (J, q) method reduces to the logarithmic method (L) (see [4]). Considering a series Σa_n such that $F(x) \equiv \Sigma_n a_n x^n$ is $e^{-(1+x)}$, it is easy to see that $F(x)$ is of boded variation over (0, 1) and thus making $\Sigma_n a_n \in |A|$. However this series is not summable (C, k) for any $k \geqslant 0$ (see [9], p. 109). Since $|A| \subset |L|$ (see [12], p. 453) not all series $|L|$ are Cesàro summable and *a fortiori* convergent. This raises the following problem:

PROBLEM. Does there exist a series which is convergent but is not summable $|L|$?

We feel that the answer will be in the affirmative. Concerning the convergence factors for series summable L we conjecture the following:

COOJECTURE. $\varepsilon_n \in (c, |L|)$ if and only if

$$\Sigma_n |\triangle \varepsilon_n| < \infty \text{ and } \Sigma_n \{|\varepsilon_n|/(n \log n)\} < \infty.$$

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