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ON THE UNIQUENESS OF THE SOLUTION OF DIRICHLET'S PROBLEM RELATIVE TO A STRONG ELLIPTIC SYSTEM OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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1. Let $\Omega \subset \mathbf{R}^2$ a bounded domain and

(1) $$\mathbf{L}\mathbf{u} = \mathbf{A}\mathbf{u}''_{xx} + 2\mathbf{B}\mathbf{u}''_{xy} + \mathbf{C}\mathbf{u}''_{yy} + \mathbf{D}\mathbf{u}'_{x} + \mathbf{F}\mathbf{u}'_{y} + \mathbf{G}\mathbf{u} = \mathbf{H}$$

a system of differential equations with second order partial derivatives with

(2) $$\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{F}, \mathbf{G} \in C(\overline{\Omega}, M_{22}(\mathbf{R})), \ \mathbf{H} \in C(\overline{\Omega}, \mathbf{R}^2),$$

and $\mathbf{u} = (u_1, u_2)$ a vectorial-function, $\mathbf{u} : \overline{\Omega} \to \mathbf{R}^2$.

Let $\mathbf{e} = (x_1, x_2) \in \mathbf{R}^2$ be so that $\mathbf{u} = |\mathbf{u}|\, \mathbf{e}$, therefore $|\mathbf{e}| = 1$.

In [5] IOAN A. RUS proved a theorem that gives conditions to assure the validity of maximum principle for the modulus of the solution of a strong elliptic nonhomogeneous system of second order partial differential equations. In essence the condition is the following:

If there exist $\alpha \in \mathbf{R}$, $\alpha \neq 0$, se that for each $\mathbf{e} \in C^2(\Omega, \mathbf{R}^n)$ we have

(3) $$\langle \mathbf{e}, \mathbf{L}\, \mathbf{e} \rangle \leqslant -\alpha^2,$$

then

(4) $$|\mathbf{u}(\mathbf{x})| \leqslant \max \left\{ \max_{x \in \partial\Omega} |\mathbf{u}(\mathbf{x})|, \ \frac{1}{\alpha^2} \max_{x \in \overline{\Omega}} |\mathbf{H}(\mathbf{x})| \right\}$$

takes place for each solution $\mathbf{u} \in C^2(\Omega, \mathbf{R}^n) \cup C(\bar{\Omega}, \mathbf{R}^n)$ of a strong elliptic system of second order equations.

The purpose of this note is to give the algebrical conditions to the coefficients of system (1) so that the relation (3) might be achieved, namely, a maximum principle might occur.

In this case the Dirichlet's boundary value problem possess at most one solution. But, since Fredholm's alternatives occur the existence of the solution is being assured, the solution exist, therefore, and yet it is unique.

Let us introduce the following notations:

$$\mathbf{e}_1 = (x_1, x_2), \quad |\mathbf{e}| = 1; \qquad \mathbf{e}_1 = (x_3, x_4) = \frac{\partial \mathbf{e}}{\partial x};$$

$$\text{(5)} \qquad \mathbf{e}_2 = (x_5, x_6) = \frac{\partial \mathbf{e}}{\partial y}; \qquad \mathbf{e}_{11} = (x_7, x_8) = \frac{\partial^2 \mathbf{e}}{\partial x^2};$$

$$\mathbf{e}_{12} = (x_9, x_{10}) = \frac{\partial^2 \mathbf{e}}{\partial x\, \partial y}; \qquad \mathbf{e}_{22} = (x_{11}, x_{12}) = \frac{\partial^2 \mathbf{e}}{\partial y^2},$$

and

$$\text{(6)} \qquad f = \langle \mathbf{e}, \mathbf{L}\,\mathbf{e} \rangle = \langle \mathbf{e}, \mathbf{A}\,\mathbf{e}_{11} \rangle + 2\langle \mathbf{e}, \mathbf{B}\mathbf{e}_{12} \rangle + \langle \mathbf{e}, \mathbf{C}\mathbf{e}_{22} \rangle + \langle \mathbf{e}, \mathbf{D}\mathbf{e}_1 \rangle + \langle \mathbf{e}, \mathbf{F}\mathbf{e}_2 \rangle + \langle \mathbf{e}, \mathbf{G}\mathbf{e} \rangle.$$

We seek for max f in the following conditions:

$$\text{(7)} \qquad \begin{aligned} &|\mathbf{e}| = 1, \ \langle \mathbf{e}, \mathbf{e}_1 \rangle = 0, \ \langle \mathbf{e}, \mathbf{e}_2 \rangle = 0, \\ &\langle \mathbf{e}, \mathbf{e}_{11} \rangle + \langle \mathbf{e}_1, \mathbf{e}_1 \rangle = 0, \ \langle \mathbf{e}, \mathbf{e}_{12} \rangle + \langle \mathbf{e}_1, \mathbf{e}_2 \rangle = 0, \\ &\langle \mathbf{e}, \mathbf{e}_{22} \rangle + \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 0. \end{aligned}$$

After having effected the calculations we conclude that:

$$\text{(6')} \qquad \begin{aligned} f = {} & g_{11}x_1^2 + (g_{12} + g_{21})x_1x_2 + d_{11}x_1x_3 + d_{12}x_1x_4 + \\ & + f_{11}x_1x_5 + f_{12}x_1x_6 + a_{11}x_1x_7 + a_{12}x_1x_8 + 2b_{11}x_1x_9 + \\ & + 2b_{12}x_1x_{10} + c_{11}x_1x_{11} + c_{12}x_1x_{12} + g_{22}x_2^2 + d_{21}x_2x_3 + \\ & + d_{22}x_2x_4 + f_{21}x_2x_5 + f_{22}x_2x_6 + a_{21}x_2x_7 + a_{22}x_2x_8 + \\ & + 2b_{21}x_2x_9 + 2b_{22}x_2x_{10} + c_{21}x_2x_{11} + c_{22}x_2x_{12}. \end{aligned}$$

Since we have a problem of connected extremum, we make of the Lagrange's multiplicators method. After long enough calculations we get the following results:

If

$$\text{(8)} \qquad P(\mathbf{A}) = P(\mathbf{B}) = P(\mathbf{C}) = 0$$

with $P(\mathbf{A}) = (a_{11} - a_{22})^2 + 4a_{12}a_{21}, \ldots$ and

$$\text{(9)} \qquad a = \frac{2\,a_{21}}{a_{11} - a_{22}} = \frac{2b_{21}}{b_{11} - b_{22}} = \frac{2c_{21}}{c_{11} - c_{22}} \neq \pm 1 \quad \text{(notation and assumtion),}$$

then

$$\text{(10)} \qquad x_1 = -ax_2, \ x_3 = Mx_2, \ x_4 = aMx_2, \ x_5 = Nx_2, \ x_6 = aNx_2,$$

where

$$x_2^2 = \frac{1}{1 + a^2}$$

$$\text{(11)} \qquad \begin{aligned} M &= \frac{\mathrm{Tr}(\mathbf{C})[d_{21} - a(d_{11} - d_{22}) - a^2d_{12}] - \mathrm{Tr}(\mathbf{B})[f_{21} - a(f_{11} - f_{22}) - a^2f_{12}]}{(1 + a^2)\,\triangle} \\ N &= \frac{\mathrm{Tr}(\mathbf{A})[f_{21} - a(f_{11} - f_{22}) - a^2f_{12}] - \mathrm{Tr}(\mathbf{B})[d_{21} - a(d_{11} - d_{22}) - a^2d_{12}]}{(1 + a^2)\,\triangle} \end{aligned}$$

while

$$\mathrm{Tr}\ (\mathbf{A}) = a_{11} + a_{22} \text{ is trace of matrix } \mathbf{A}, \ldots$$

$$\triangle = \mathrm{Tr}(\mathbf{A})\mathrm{Tr}(\mathbf{C}) - \mathrm{Tr}(\mathbf{B})^2 \neq 0 \ \text{(assumption).}$$

For Lagrange's multiplicators we get the following expressions:

$$\text{(13)} \qquad \begin{aligned} \lambda_1 = {} & \frac{g_{22} - a^2g_{11}}{a^2 - 1} + a\,\frac{M(d_{22} - d_{11}) + N(f_{22} - f_{11})}{2(a^2 + 1)} + \\ & + a^2\,\frac{d_{12} + d_{21} + f_{12} + f_{21}}{a^4 - 1} + \frac{M(d_{12}a^2 + d_{21}) + N(f_{12}a^2 + f_{21})}{2(a^2 - 1)} - \\ & - (1 + a^2)\,\frac{M^2(a_{22} - a_{11}) + 2MN(b_{22} - b_{11}) + N^2(c_{22} - c_{11})}{2(a^2 - 1)}, \end{aligned}$$

$$\lambda_2 = \frac{a(d_{12} + d_{21}) - a^2d_{11} - d_{22}}{1 + a^2}, \quad \lambda_3 = \frac{a(f_{12} + f_{21}) - a^2f_{11} - f_{22}}{1 + a^2},$$

$$\lambda_4 = -\frac{\mathrm{Tr}(\mathbf{A})}{2}, \quad \lambda_5 = -\mathrm{Tr}(\mathbf{B}), \quad \lambda_6 = -\frac{\mathrm{Tr}(\mathbf{C})}{2}$$

Noting by f^* the value of f, we get

$$\text{(14)} \qquad \begin{aligned} f^* = {} & M^2(aa_{12} - a_{22}) - 2MN(ab_{12} - b_{22}) + N^2(ac_{12} - c_{22}) + \\ & + \frac{M[d_{21} + a(d_{22} - d_{11}) - a^2d_{12}] + N[f_{21} + a(f_{22} - f_{11}) - a^2f_{12}]}{1 + a^2} + \\ & + \frac{g_{22} - a(g_{12} + g_{21}) + a^2g_{11}}{1 + a^2}. \end{aligned}$$

Now we suppose that

$$g_{11} < 0\,; \quad 4g_{11}g_{22} - (g_{12} + g_{21})^2 > 0\,;$$

$$d_{11}d_{21}(g_{12} + g_{21}) - d_{11}^2 g_{22} - d_{21}^2 g_{11} < 0\,; \tag{15}$$

$$d_{11}d_{22} - d_{21}d_{12} \neq 0 \text{ (that is, } \mathbf{D} \text{ is nondegenerated matrix).}$$

Remarks : 1) The conditions (8), (9) and (12) play the role of simplifying the calculations that appear and permit an effective expression of the variables.

2) Conditions (15) assure (see, for example [1]) that the quadratic form f is negatively defined, that is its maximum is nonpositive.

We have the following theorem.

THEOREM 1. *If* i) *system* (1) *is strong elliptic in domain* Ω,

ii) *matrices* **A**, **B**, **C** *satisfy the relations* (8), (9), (12),

iii) *matrices* **D**, **G** *satisfy the relations* (15),

iv) *there is* $\alpha \in \mathbf{R}$, $\alpha \neq 0$, so that $f^* \leqslant -\alpha^2$, *then*

$$|\mathbf{u}(\mathbf{x})| \leqslant \max\left\{\max_{x\in\partial\Omega} |\mathbf{u}(\mathbf{x})|, \frac{1}{\alpha^2}\max_{x\in\bar{\Omega}} |\mathbf{H}(\mathbf{x})|\right\}.$$

Remarks 3) For the strong elliptic systems of the form (1) the Theorem 1 gives conditions for the validity of the maximum principle, only relative to the coefficients of the system, thus eliminating that any unit vector **e** which appears in the condition (3).

4) The Theorem 1 leads easily to uniqueness of the solution of Dirichlet's problem

$$\left.\begin{aligned} \mathbf{Lu} &= \mathbf{H} \text{ in } \Omega \\ \mathbf{u} &\in \mathbf{C}^2(\bar{\Omega}, \mathbf{R}^2) \\ \mathbf{u} &= \mathbf{h} \text{ on } \partial\Omega \end{aligned}\right\} \tag{16}$$

where $\mathbf{h} \in C(\partial\Omega, \mathbf{R}^2)$.

It really takes place

THEOREM 2. *Under the conditions of Theorem* 1, *from*

$$\left.\begin{aligned} \mathbf{Lu} &= \mathbf{0} \text{ in } \Omega \\ \mathbf{u} &\in C^2(\bar{\Omega}, \mathbf{R}^2) \\ \mathbf{u} &= \mathbf{0} \text{ on } \partial\Omega \end{aligned}\right\} \Rightarrow \mathbf{u} \equiv \mathbf{0} \text{ in } \Omega. \tag{17}$$

Proof. The demonstration is obviously as basis Theorem 1 within conditions (17).

2. *Example.* Let be system (1) in which

$$\mathbf{A} = \begin{bmatrix} 5 & -1/2 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 10 & -2 \\ 8 & 2 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 3 & 1 \\ -5 & 1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} -1 & 3 \\ -1 & -2 \end{bmatrix} \tag{18}$$

We conclude that

$$a = 2, \quad \triangle = 92, \quad M = -43/115, \quad N = -9/46,$$

$$f^* = -20473/26450. \tag{19}$$

We can easily find out that the conditions of Theorem 1 are achieved. We have therefore the following theorems

THEOREM 3. *If the coefficients of system* (1) *are given by relations* (18) *and* $\alpha = 0{,}8797$, *then*

$$|\mathbf{u}(\mathbf{x})| = \max\left\{\max_{x\in\partial\Omega} |\mathbf{u}(\mathbf{x})|, \frac{1}{0{,}77\,387}\max_{x\in\bar{\Omega}} |\mathbf{H}(\mathbf{x})|\right\},$$

where $\mathbf{u} \in C^2(\Omega, \mathbf{R}^2) \cap C(\bar{\Omega}, \mathbf{R}^2)$ is the solution of system (1).

THEOREM 4. *Under the conditions of Theorem* 3, *Dirichlet's problem has an unique solution.*

Remark 5) In [2] C. MIRANDA gives theorems of maximum for the solutions of elliptic systems of second order equations where the matrices of the terms with second order derivatives are diagonal, while the respective conditions are expressed by a certain vector n-dimensional. In a next work, we intend to improve Miranda's results in the case of nondiagonal matrices, and concerning the conditions, to be given only relative to the coefficients.

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