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A NOTE ON THE CONVERGENCE OF STEFFENSEN'S METHOD

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Let X be a real linear Banach space and $P : X \to X$ a continuous mapping. We shall note respectively by $[x', x''; P]$ and $[x', x'', x''';P]$ the symmetrical divided difference of the mapping P for the points x', x'', $x''' \in X$ [2], [3]. Let's consider the equation

$$(1) \qquad P(x) = x - \Phi(x) = 0.$$

If (x_n) is a sequence of the space X, then (u_n) will denote the sequence defined by $u_n = \Phi(x_n)$ and Γ_n will denote the inverse of the linear mapping $[x_n, u_n; P]$, if it exists. In our paper [1] (Theorem 4.1) we gave a sufficient condition of the existence and the approximation for the roots of equation (1). The approximating sequence was defined there by

$$(2) \qquad x_{n+1} = x_n - \Gamma_n P(x_n), \; n = 0, 1, 2, \ldots$$

In this paper we shall give a better version of the theorem we were talking about. By better version we mean that the mapping $[x', x''; P]$ must not be bounded and that the radius of the ball $S(x_0, r)$, where the bilinear mapping $[x', x'', x'''; P]$ has to be bounded, doesn't depend on the upper bound of this mapping.

The next theorem gives sufficient conditions of the existence and the approximation for the roots of equation (1):

THEOREM. *We suppose that there exists a point* $x_0 \in X$ *and the constants* B_0, η_0, K *so that the following conditions are satisfied:*

1° $||P(x_0)|| = ||x_0 - \Phi(x_0)|| = ||x_0 - u_0|| \leqslant \eta_0$;

2° *for* x_0 *and* $u_0 = \Phi(x_0)$ *there exists the inverse of the divided difference* $[x_0, u_0; P]$ and $\|\Gamma_0\| = \|[x_0, u_0; P]^{-1}\| \leqslant B_0$;

3° $\sup \{\|[x', x'', x'''; P]\| : x', x'', x''' \in S(x_0, r)\} \leqslant K$;

4° $h_0 = B_0 K \left(\frac{4}{3} + 2B_0\right) \eta_0 \leqslant \frac{1}{3}$, *where*

$$S = \{x \in X : \|x - x_0\| < r\}, \quad r = 2B_0\eta_0 + \eta_0.$$

In these conditions the equality

$$x_{n+1} = x_n - \Gamma_n P(x_n), \quad n = 0, 1, 2, \ldots \tag{2}$$

defines by recurence a sequence (x_n) *having the following qualities:*

(i) $x^* = \lim_{n\to\infty} x_n$, $x^* \in \bar{S}$ *exists and* x^* *is the solution of equation* (1);

(ii) *the rate of convergence is given by*

$$\|x_n - x^*\| \leqslant 4B_0\eta_0 \left(\frac{2}{3}\right)^n \left(\frac{3}{4}\right)^{2^n} (3h_0)^{2^n - 1}.$$

Proof. By condition 2°, using equality (2) we construct x_1. From (1) and (2), by the meen of 1° and 2° we obtain

$$\|x_0 - u_0\| = \|x_0 - \Phi(x_0)\| = \|P(x_0)\| \leqslant \eta_0,$$

$$\|x_0 - x_1\| = \|\Gamma_0 P(x_0)\| \leqslant B_0\eta_0.$$

$$\|x_1 - u_0\| = \|x_0 - [x_0, u_0; P]^{-1} P(x_0) - u_0\| = \tag{3}$$

$$= \|P(x_0) + [x_0, u_0; P]^{-1} P(x_0)\| \leqslant \eta_0(1 + B_0).$$

From formula

$$P(z) = P(x) + [x, y; P](z - x) + [z, x, y; P](z - y)(z - x), \tag{4}$$

which is true for all $x, y, z \in X$, for $z = x_1$, $x = x_0$, $y = u_0$, using (2), we get

$$P(x_1) = [x_1, x_0, u_0; P](x_1 - u_0)(x_1 - x_0) \text{ and}$$

$$\|P(x_1)\| \leqslant K\|x_1 - x_0\| \cdot \|x_1 - u_0\| \leqslant KB_0\eta_0^2(1 + B_0) \leqslant h_0\eta_0 = \tag{5}$$

$$= \eta_1 < \frac{1}{3}\eta_0.$$

Further we have

$$\|x_0 - u_1\| = \|x_0 - x_1 + x_1 - u_1\| \leqslant \|x_0 - x_1\| + \|P(x_1)\| \leqslant \tag{6}$$

$$\leqslant B_0\eta_0 + \frac{1}{3}\eta_0 = \eta_0\left(B_0 + \frac{1}{3}\right).$$

From 1° results that the term x_1 of (x_n) can be constructed using the equality (2) and thus $u_1 = \Phi(x_1)$ can also be obtained. We now check

conditions 1° — 4° for the points x_1, u_1 with the constants B_1, η_1 and K.

From identity

$$[x,y;P] - [z,v;P] = \tag{7}$$

$$= [x,y;P] - [x,z;P] + [x,z;P] - [z,v;P] =$$

$$= [x,y,z;P](y - z) + [x,z,v;P](x - v),$$

which is true for all $x, y, z, v \in X$, taking $x = x_0$, $y = u_0$, $z = x_1$, $v = u_1$, we get

$$\|\Gamma_0\{[x_0,u_0;P] - [x_1,u_1;P]\}\| \leqslant KB_0(\|x_1 - u_0\| + \|x_0 - u_1\|) \leqslant \tag{8}$$

$$\leqslant KB_0\left[\eta_0(1 + B_0) + \eta_0\left(B_0 + \frac{1}{3}\right)\right] = B_0K\left(\frac{4}{3} + 2B_0\right)\eta_0 = h_0 \leqslant \frac{1}{3} (<1)$$

Here we used (3) and (6) and the fact that $x_1, u_1, u_0 \in S(x_0, r)$.

If I denotes the identity operator of X, then we can write

$$\{\Gamma_0[x_1,u_1;P]\}^{-1} = \{I - \Gamma_0\{[x_0,u_0;P] - [x_1,u_1;P]\}\}^{-1},$$

which by (8) leads to the existence of the mapping $\{\Gamma_0[x_1,u_1;P]\}^{-1}$ and to the inequality

$$\|\{\Gamma_0[x_1, u_1; P]\}^{-1}\| \leqslant \frac{1}{1 - h_0}.$$

Using the evident equality $\{\Gamma_0[x_1, u_1; P]\}^{-1}\Gamma_0 = \Gamma_1$, we obtain

$$\|\Gamma_1\| \leqslant \frac{B_0}{1 - h_0} = B_1 \leqslant \frac{3}{2}B_0 \qquad (B_0 < B_1).$$

Thus condition 2° is satisfied for the points x_1, u_1. Condition 1° for these points is verified in (5). We have

$$h_1 = B_1K\left(\frac{4}{3} + 2B_1\right)\eta_1 = \frac{B_0}{1 - h_0}K\left(\frac{4}{3} + \frac{2B_0}{1 - h_0}\right)h_0\eta_0 <$$

$$< h_0\frac{B_0K\left(\frac{4}{3} + 2B_0\right)\eta_0}{(1 - h_0)^2} = \frac{h_0^2}{(1 - h_0)^2} \leqslant \frac{1}{4} < \frac{1}{3}.$$

and so 4° is satisfied with the constants B_1, η_1 and K.

By mathematical induction we shall prove the followings:

a) $x_n \in S(x_0, r)$,

b) $\|P(x_n)\| \leqslant h_{n-1}\eta_{n-1} = \eta_n \leqslant \frac{\eta_0}{3^n}$,

c) $u_n \in S(x_0, r)$,

d) $\Gamma_n = [x_n, u_n; P]^{-1}$ exists and $\|\Gamma_n\| \leqslant \frac{B_{n-1}}{1 - h_{n-1}} = B_n \leqslant \left(\frac{3}{2}\right)^n B_0$,

e) $h_n = B_n K\left(\frac{4}{3} + 2B_n\right) \eta_n \leqslant \frac{h_{n-1}^2}{(1-h_{n-1})^2} \leqslant \left(\frac{3}{2}\right)^2 h_{n-1}^2 < \frac{1}{3}$, for $n=1,2,3,\ldots$

We already checked a) —d) for $n = 1$.

Let's suppose that they are true for all $k \leqslant n$, where $n > 1$.

From c) results that x_{n+1} can be constructed using equality (2). From b) and d) by (1) and (2) it results $||x_{n+1} - x_n|| \leqslant B_n\eta_n \leqslant \frac{B_0\eta_0}{2^n}$,

$$(9)\quad ||x_{n+1} - u_n|| = ||x_n - \Gamma_n P(x_n) - u_n|| = ||x_n - u_n - [x_n, u_n; P]^{-1}P(x_n)|| =$$
$$= ||x_n - \Phi(x_n) - [x_n, u_u; P]^{-1}P(x_n)|| = ||P(x_n) +$$
$$+ [x_n, u_n; P]^{-1} P(x_n)|| \leqslant \eta_n(1 + B_n).$$

We further have

$$||x_{n+1} - x_0|| \leqslant ||x_{n+1} - x_n|| + ||x_n - x_{n-1}|| + \ldots + ||x_1 - x_0|| \leqslant$$
$$\leqslant B_0\eta_0\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}\right) < 2B_0\eta_0.$$

This means that $x_{n+1} \in S(x_0, r)$ and so a) is proved for $k = n + 1$. From (4), taking $z = x_{n+1}$, $x = x_n$, $y = u_n$, and using (2), we obtain

$$P(x_{n+1}) = [x_{n+1}, x_n, u_n; P](x_{n+1} - u_n)(x_{n+1} - x_n) \text{ and}$$
$$||P(x_{n+1})|| \leqslant KB_n\eta_n^2(1 + B_n) \leqslant h_n\eta_n = \eta_{n+1} < \frac{1}{4}\frac{\eta_0}{3^n} < \frac{\eta_0}{3^{n+1}}.$$

Thus b) is also true for $k = n + 1$.

$$||x_0 - u_{n+1}|| \leqslant ||x_0 - x_1|| + ||x_1 - x_2|| + \ldots + ||x_n - x_{n+1}|| +$$
$$+ ||x_{n+1} - u_{n+1}|| \leqslant \eta_0 B_0\left(1 + \frac{1}{2} + \ldots + \frac{1}{2^n}\right) + ||P(x_{n+1})|| \leqslant$$
$$\leqslant 2B_0\eta_0 + \frac{\eta_0}{3^{n+1}},$$

means $u_{n+1} \in S(x_0, r)$, which is c) for $k = n + 1$.

$$(10)\quad ||x_n - u_{n+1}|| \leqslant ||x_n - x_{n+1}|| + ||x_{n+1} - u_{n+1}|| \leqslant B_n\eta_n + \eta_{n+1} =$$
$$= \eta_n\left(B_n + \frac{1}{3}\right).$$

If we put $x = x_n$, $y = u_n$, $z = x_{n+1}$, $v = u_{n+1}$, then from (7), using (9) and (10) and the fact that $x_n, u_n, x_{n+1}, u_{n+1} \in S(x_0, r)$ we get

$$(11)\quad ||\Gamma_n\{[x_n, u_n; P] - [x_{n+1}, u_{n+1}; P]\}|| \leqslant KB_n(||u_n - x_{n+1}|| +$$
$$+ ||x_n - u_{n+1}||) \leqslant B_n K\left(\frac{4}{3} + 2B_n\right)\eta_n = h_n < \frac{1}{3} \quad (<1).$$

The obvious equality

$$\{\Gamma_n[x_{n+1}, u_{n+1}; P]\}^{-1} = \{I - \{\Gamma_n[x_n, u_n; P] - \Gamma_n[x_{n+1}, u_{n+1}; P]\}\}^{-1},$$

together with (11) leads us to the existence of $\{\Gamma_n[x_{n+1}, u_{n+1}; P]\}^{-1}$, and so to the existence of $\{\Gamma_n[x_{n+1}, u_{n+1}; P]\}^{-1}\Gamma_n = \Gamma_{n+1}$. Thus we have

$$||\Gamma_{n+1}|| \leqslant \frac{B_n}{1 - h_n} = B_{n+1} \leqslant \left(\frac{3}{2}\right)^{n+1} B_0,$$

which means that d) is proved for $k = n + 1$.

The relations

$$h_{n+1} = B_{n+1} K\left(\frac{4}{3} + 2B_{n+1}\right)\eta_{n+1} = \frac{B_n K}{1 - h_n}\left(\frac{4}{3} + \frac{2B_n}{1 - h_n}\right) h_n\eta_n <$$
$$< \frac{B_n K\left(\frac{4}{3} + 2B_n\right)}{(1 - h_n)^2}\eta_n h_n = \frac{h_n^2}{(1 - h_n)^2} < \left(\frac{3}{2}\right)^2 h_n^2 \leqslant \frac{1}{3}$$

mean that e) is also true for $k = n + 1$.

The expressions e) and b) lead to

(12)
$$h_n \leqslant \left(\frac{3}{2}\right)^2 h_{n-1}^2 \leqslant \left(\frac{3}{2}\right)^2 \left[\left(\frac{3}{2}\right)^2 h_{n-2}^2\right]^2 \leqslant \ldots \leqslant \left(\frac{3}{2}\right)^{2(2^n - 1)} h_0^{2^n}, \quad n = 1, 2, \ldots,$$
$$\eta_n = \leqslant h_{n-1}\eta_{n-1} \leqslant h_{n-1} h_{n-2} h_{n-3} \ldots h_0\eta_0 =$$
$$= \eta_0 h_0 h_0^2 h_0^{2^2} \ldots h_0^{2^{n-1}} \left(\frac{3}{2}\right)^{2(2-1)} \left(\frac{3}{2}\right)^{2(2^2-1)} \ldots \left(\frac{3}{2}\right)^{2(2^{n-1}-1)} =$$
$$= \eta_0 h_0^{2^n - 1}\left(\frac{3}{2}\right)^{2(2^n - 2) - 2(n-1)} = 3\eta_0\left(\frac{4}{9}\right)^{n+1}\left(\frac{3}{4}\right)^{2^n}(3h_0)^{2^n - 1}.$$

On the basis of the inequality $||x_{n+1} - x_n|| \leqslant B_n\eta_n$, using d) and (12), we obtain

$$||x_n - x_{n+p}|| \leqslant B_n\eta_n + B_{n+1}\eta_{n+1} + \ldots + B_{n+p-1}\eta_{n+p-1} \leqslant$$
$$\leqslant \frac{4}{3} B_0\eta_0\left(\frac{2}{3}\right)^n\left(\frac{3}{4}\right)^{2^n}(3h_0)^{2^n - 1}\sum_{i=1}^{p}\left(\frac{2}{3}\right)^{i-1}\left(\frac{9h_0}{4}\right)^{2^n(2^{i-1}-1)} <$$
$$< 4B_0\eta_0\left(\frac{2}{3}\right)^n\left(\frac{3}{4}\right)^{2^n}(3h_0)^{2^n - 1},$$

which means that the sequence (x_n) has the limit $x^* \in \bar{S}(x_0, r)$, because X is a Banach space.

Now we only have to prove that x^* is a solution of (1).

The evident equality

$$[x_n, u_n; P] = [x_n, u_n, x^*; P](u_n - x^*) + [x_n, x^*, x_0; P](x_n - x_0) + [x^*, x_0; P],$$

together with condition 3° of the Theorem leads to

$$\text{(13)} \qquad |||[x_n, u_n ; P]||| \leqslant 3Kr + |||[x^*, x_0 ;P]||| = M.$$

which means that the linear mapping $[x_n, u_n ;P]$ is bounded.

By (2), using (13) we can write

$$||P(x_n)|| = ||[x_n, u_n ;P](x_{n+1} - x_n)|| \leqslant M||x_{n+1} - x ||,$$

which for $n \to \infty$ gives

$$\lim_{n\to\infty} ||P(x_n)|| = ||P(x^*)|| = 0.$$

So $P(x^*) = 0$, which shows that x^* is a solution of equation (1).

Exemple. Let's consider the equation [4]

$$\text{(14)} \qquad P(x) = x - \Phi(x) = x^3 - 2x - 5 = x - (-x^3 + 3x + 5) = 0.$$

If $x_0 = 2{,}1$, then $P(x_0) = 0{,}061$, $u_0 = \Phi(x_0) = 2{,}039$ and $||\Gamma_0|| \leqslant 0{,}093$. Thus $||P(x_0)|| = 0{,}061 = \eta_0$, $||\Gamma_0|| \leqslant 0{,}093 = B_0$ and $r = 2B_0\eta_0 + \eta_0 = 2 \cdot 0{,}093 \cdot 0{,}061 + 0{,}061 = 0{,}072346$. We have

$$[x', x'' ; P] = x'^2 + x' x'' + x''^2 - 2 \text{ and } [x', x'', x''' ;P] = x' + x'' + x'''$$

So $\sup \{|||[x',x'',x''' ;P]||| : x',x'',x''' \in [2{,}027654 ; 2{,}172346]\} \leqslant 6{,}517038.$

$$h_0 = B_0K\left(\frac{4}{3} + 2B_0\right)\eta_0 \leqslant 0{,}093 \cdot 6{,}52(1{,}334 + 0{,}186) \cdot 0{,}061 =$$

$$= 0{,}0562216992 < \frac{1}{3}.$$

The conditions of the Theorem being satisfied the equation (14) has a solution in $[2{,}027654 ; 2{,}172346]$, to which the sequence defined by (2) converges.

REFERENCES

[1] Balázs, M., *Contribution to the Study of Solving the Equations in Banach Spaces.* Doctor Thesis, Cluj (1969).

[2] Balázs, M. Goldner, G., *Diferențe divizate în spații Banach și unele aplicații ale lor* (Roumanian). Stud. și Cerc. Matematice, **7**, *21*, 985–996 (1969).

[3] Ulm, S., *On Generalized Divided Differences*, I–II (Russian). Izv. Akad. Nauk. ESSR, **16**, 13–26 and 146–155 (1967).

[4] Werner, W., *Über ein iteratives Verfahren der Ordnung* $1 + \sqrt{2}$ *zur Nullstellenbestimmung.* ZAMM, Band **59**, Heft 3, T. 86–87 (1979).

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